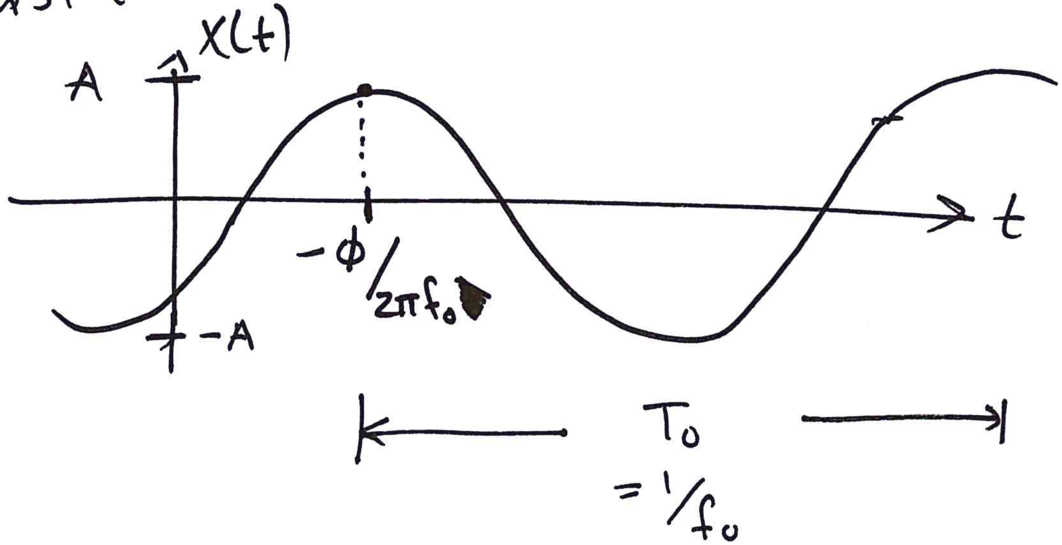
Lecture 3

9/14/16

Last time: $x(t) = A\cos(2\pi f_0 t + \phi)$



$\cos(0) = 1$

$\cos(2\pi f_0 t + \phi) = 1$

$\Updownarrow$

$2\pi f_0 t + \phi = 0$

$t = \dfrac{-\phi}{2\pi f_0} \quad (+2\pi)$

Today: Complex Representations of Sinusoids

+ adding sinusoids of the same frequency

e.g. $x(t) = 3\cos(2\pi\, \underline{4t} - \pi/2) - 5\cos(2\pi\, \underline{4t} + \pi/6)$

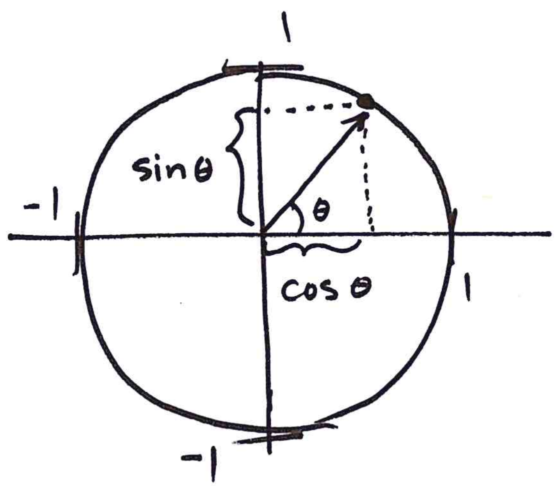
Option 1 : use trig identities

- tedious
- not linear — not ~~at~~ just addition and subtraction

$\Rightarrow$ solving more sophiticated problems much more complicated

Option 2 : use complex numbers — make analysis + computation much easier.

Recall unit circle



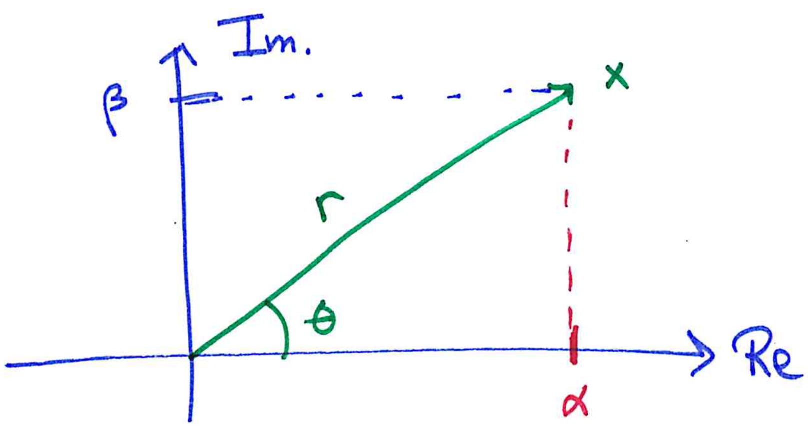
Review complex numbers:

$$x = \alpha + j\beta$$

$\alpha = \mathrm{Re}\{x\}$ = real part

$\beta = \mathrm{Im}\{x\}$ = imaginery part

$j = \sqrt{-1}$



$$r = \sqrt{\alpha^2 + \beta^2}$$

$$\theta = \cancel{}\, \tan^{-1}(\beta/\alpha)$$

$\Rightarrow$ $\underline{\underline{x = re^{j\theta}}}$

$\alpha = r\cos\theta$

$\beta = r\sin\theta$

Ex 1 : $x = \underline{3 - 4j}$

$\mathrm{Re}\{x\} = 3$, $\mathrm{Im}\{x\} = -4$

$r = 5$ $\quad \theta = \tan^{-1}(-4/3)$

$x = \underline{5e^{j\theta}}$

Ex 2 : $x = (3-4j)^3$

$= (5e^{j\theta})^3$

$= 125e^{j3\theta}$

$\mathrm{Re}\{x\} = 125\cos(3\theta)$

$\mathrm{Im}\{x\} = 125\sin(3\theta)$

Euler's Identity

$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

$$\Rightarrow \cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2} = \frac{(\cos(\theta) + j\sin\theta) + (\cos(\theta) - j\sin\theta)}{2}$$

$$= \cos(\theta) + 0$$

$$\sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j} = \frac{(\cos\theta + j\sin\theta) + (-\cos\theta + j\sin\theta)}{2j}$$

$$= \sin(\theta)$$

$$\mathrm{Re}(e^{j\theta}) = \cos(\theta), \quad \mathrm{Im}(e^{j\theta}) = \sin\theta$$

How does this help?

Recall trig identity: $\cos x \cdot \cos y = \frac{1}{2}\left(\cos(x+y) + \cos(x-y)\right)$

$$\cos x \cdot \cos y = \frac{e^{jx} + e^{-jx}}{2} \cdot \frac{e^{jy} + e^{-jy}}{2}$$

$$= \frac{1}{4}\left(e^{jx} \cdot e^{jy} + e^{-jx} e^{jy} + e^{jx} e^{-jy} + e^{-jx} e^{-jy}\right)$$

$$= \frac{1}{2}\left(\frac{e^{j(x+y)} + e^{-j(x-y)} + e^{+j(x-y)} + e^{-j(x+y)}}{2}\right)$$

$$= \frac{1}{2}\left(\cos(x+y) + \cos(x-y)\right)$$

Addition of sinusoids w/ same frequency:

$$x(t) = \sum_{k=1}^{N} A_k \cos(\underset{\omega_0}{\cancel{2\pi f_0}}\, t + \phi_k)$$ — same freq, diff Amps, phases

$$= \sum_{k=1}^{N} \mathrm{Re}\left\{ A_k e^{j(\omega_0 t + \phi_k)} \right\}$$

$$= \mathrm{Re}\left\{ \sum_{k=1}^{N} A_k e^{j(\omega_0 t + \phi_k)} \right\}$$

$$= \mathrm{Re}\left\{ e^{j\omega_0 t} \underbrace{\sum_{k=1}^{N} A_k e^{j\phi_k}}_{\substack{\text{this is complex \#,} \\ \text{we can write as} \\ A e^{j\phi}}} \right\} = \mathrm{Re}\left\{ e^{j\omega_0 t} \cdot A e^{j\phi} \right\}$$

$$= \mathrm{Re}\left\{ A e^{j(\omega_0 t + \phi)} \right\}$$

$$= A\cos(\omega_0 t + \phi)$$

Ex :

$$x(t) = 3\cos(2\pi 4t - \pi/2) - 5\cos(2\pi 4t + \pi/3)$$

$$= Re\{3e^{j(2\pi 4t - \pi/2)} - 5e^{j(2\pi 4t + \pi/3)}\}$$

$$= Re\{\underset{(a)}{e^{j2\pi 4t}}(\underbrace{3e^{-j\pi/2}}_{\alpha_1 + j\beta_1} \underbrace{- 5e^{j\pi/3}}_{+ \quad \alpha_2 + j\beta_2})\}$$

$$\Rightarrow \alpha_3 + j\beta_3$$

$$= -2.5 - 7.33j$$

$$= \underset{(b)}{7.74 . e^{-j1.90}}$$

$$= Re\{7.74e^{j(2\pi 4t - 1.90)}\}$$

$$= 7.74\cos(2\pi 4t - 1.90)$$