Lecture 5

$e^{j\theta} = \cos(\theta) + j\sin(\theta)$

$\cos(\theta) = \frac{1}{2} e^{j\theta} + \frac{1}{2} e^{-j\theta}$

The spectrum of a signal is the set of frequencies and their complex amplitudes making up the signal.

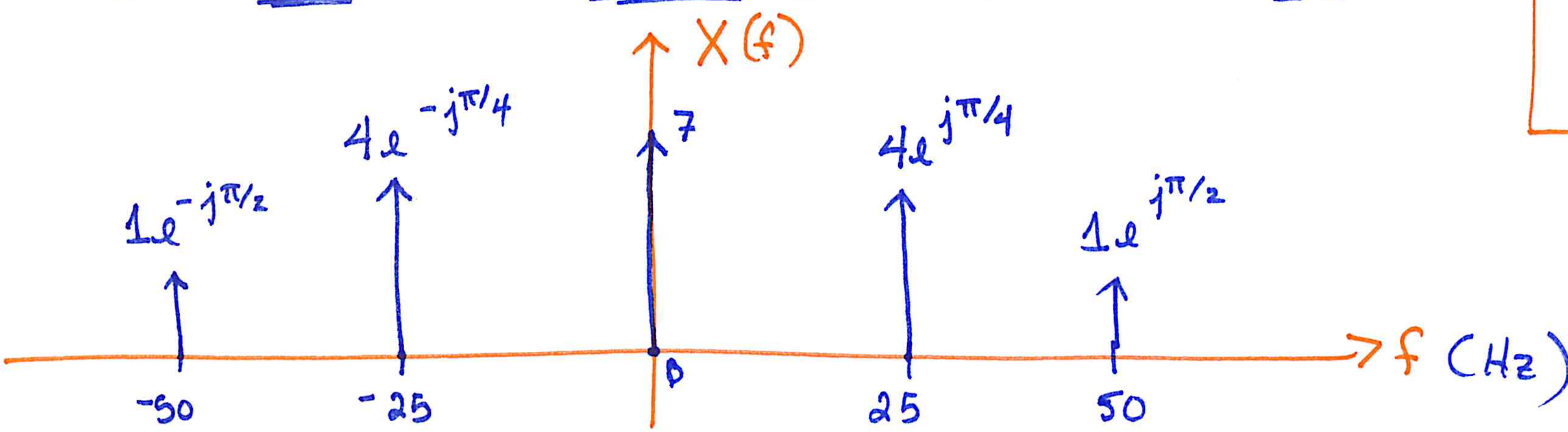
Ex 1: $x(t) = 7 + 8\cos(50\pi t + \pi/4) - 2\cos(100\pi t - \pi/2)$

$x(t) = 7 + 8\cos(50\pi t + \pi/4) + 2\cos(100\pi t - \pi/2 + \pi)$

$= 7 + 4e^{j50\pi t} e^{j\pi/4} + 4e^{-j50\pi t} e^{-j\pi/4} + 1e^{j100\pi t + \frac{\pi}{2}j} + 1e^{-j100\pi t - \frac{\pi}{2}j}$

remember:

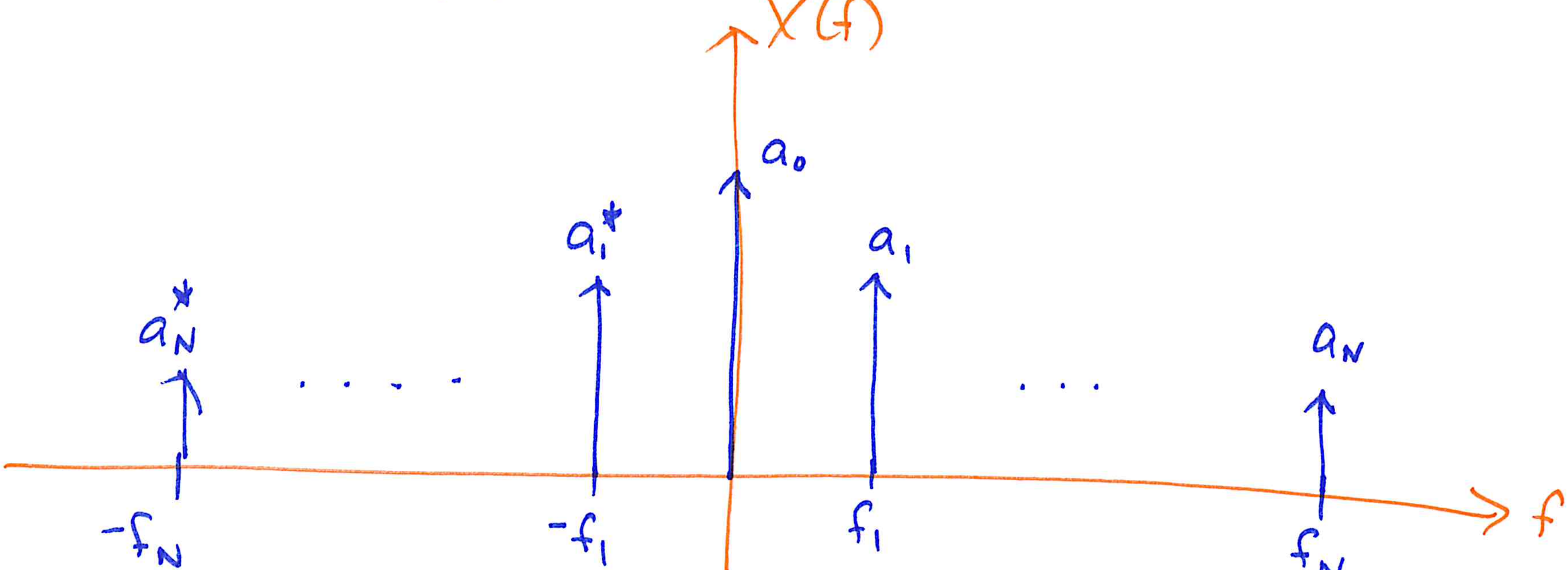
$-\cos(\theta) = \cos(\theta + \pi)$



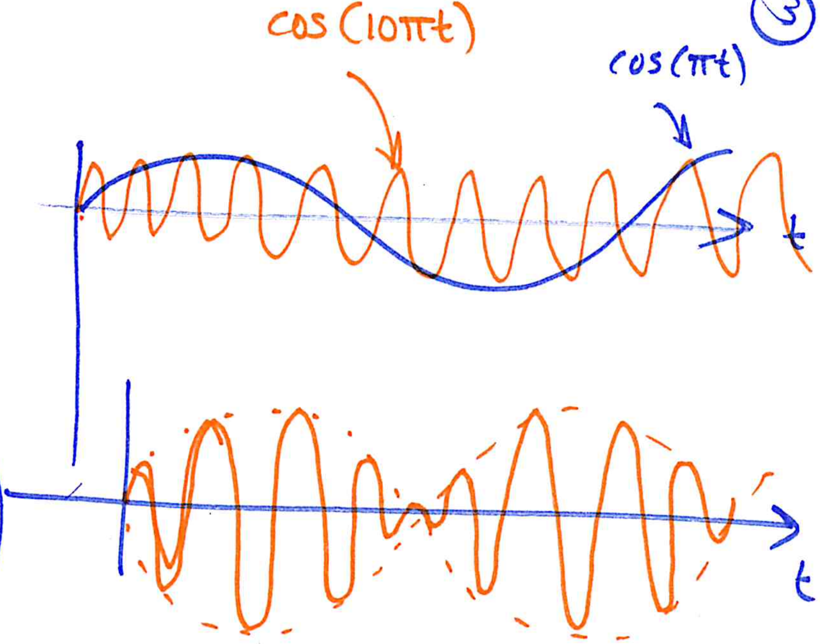
{ Any signal $x(t)$ can be almost perfectly represented by a sum of N sinusoids, provided N is large enough! } !

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi f_k t + \phi_k)$$

$$= a_0 + \sum_{k=1}^{N} \left(a_k e^{j2\pi f_k t} + a_k^* e^{-j2\pi f_k t} \right)$$



Multiplication of Sinusoids



$$x(t) = \cos(\pi t)\cdot\cos(10\pi t)$$

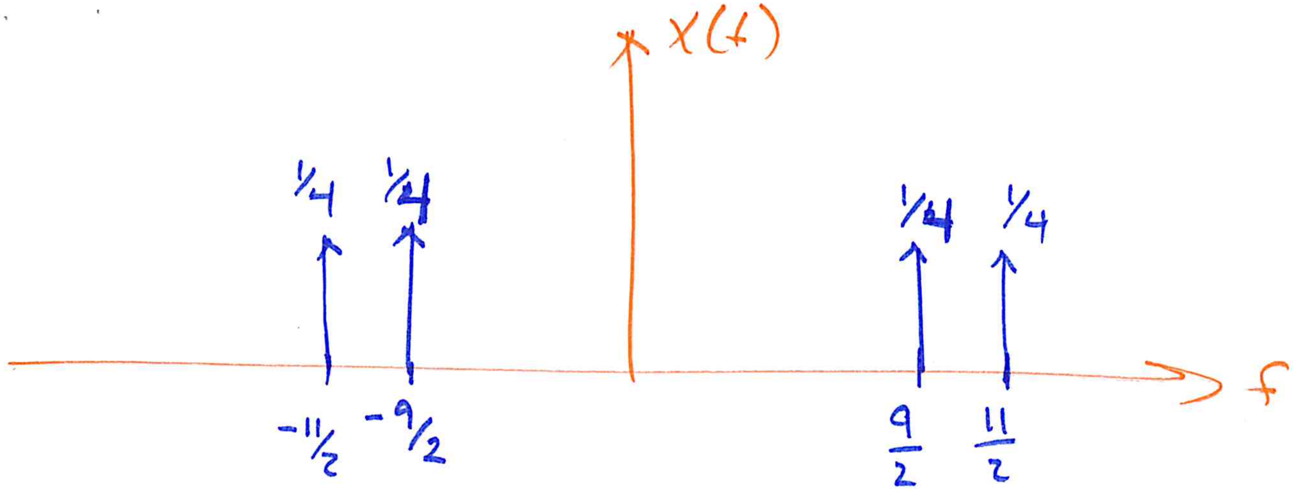
what is the spectrum of $x(t)$?

$$x(t) = \left(\frac{1}{2}e^{j\pi t} + \frac{1}{2}e^{-j\pi t}\right)\left(\frac{1}{2}e^{j10\pi t} + \frac{1}{2}e^{-j10\pi t}\right)$$

$$= \frac{1}{4}e^{j\pi t}e^{j10\pi t} + \frac{1}{4}e^{-j\pi t}e^{j10\pi t} + \frac{1}{4}e^{j\pi t}e^{-j10\pi t} + \frac{1}{4}e^{-j\pi t}e^{-j10\pi t}$$

$$= \frac{1}{2}\left(\frac{1}{2}e^{j11\pi t} + \frac{1}{2}e^{j9\pi t} + \frac{1}{2}e^{-j9\pi t} + \frac{1}{2}e^{-j11\pi t}\right)$$

$$= \frac{1}{2}\left(\cos(11\pi t) + \cos(9\pi t)\right)$$



Generally:

if $x(t) = \cos(2\pi f_c t) \cdot \cos(2\pi f_k t)$

$$= \frac{1}{2}\cos(2\pi(f_c + f_k)t) + \frac{1}{2}\cos(2\pi(f_c - f_k)t)$$

Application: Amplitude Modulation (AM Radio)

$$v(t) = \text{speech/music / acoustic signal}$$
$$\approx \sum_{k=1}^{N} \left(a_k e^{j2\pi f_k t} + a_k^* e^{-j2\pi f_k t} \right)$$

where f_k's $\lesssim \sim 15$ kHz

this low-frequency signal does not propagate well across long distances.

Solution: multiply $v(t)$ by higher frequency sinusoid!
(Reginald Fessenden, 1900)

AM signal:

$$x(t) = v(t) \cdot \underbrace{\cos(2\pi f_c t)}_{\text{"carrier signal"}}$$

f_c = "carrier frequency"

AM signals

$$v(t) = \sum_{k=1}^{N} \left(a_k e^{j2\pi f_k t} + a_k^* e^{-j2\pi f_k t} \right)$$

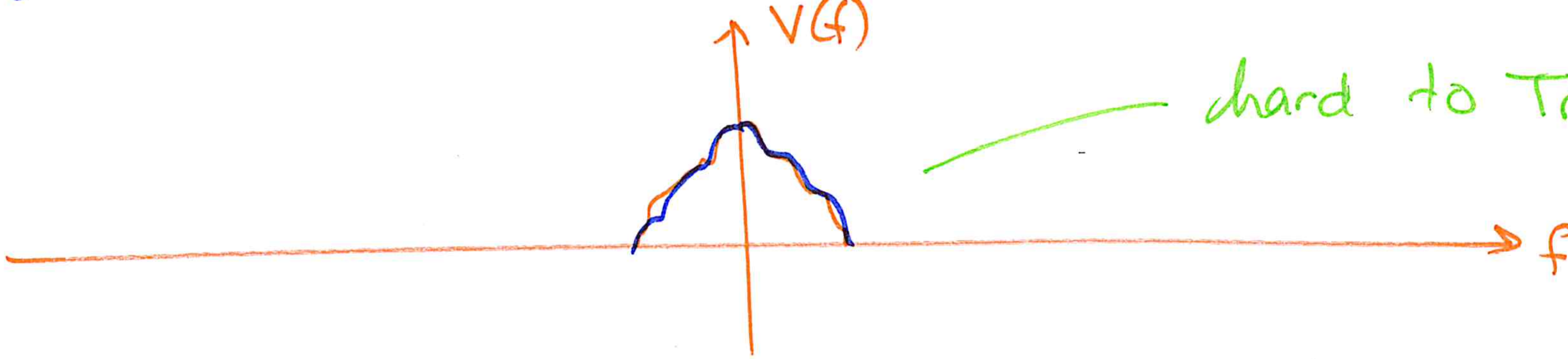
$$\text{carrier}(t) = \cos(2\pi f_c t) = \frac{1}{2} e^{j2\pi f_c t} + \frac{1}{2} e^{-j2\pi f_c t}$$

$$x(t) = v(t)\,\text{carrier}(t)$$

$$= \left(\frac{1}{2} e^{j2\pi f_c t} + \frac{1}{2} e^{-j2\pi f_c t} \right) \sum_{k=1}^{N} \left(a_k e^{j2\pi f_k t} + a_k^* e^{-j2\pi f_k t} \right)$$

$$= \sum_{k=1}^{N} \frac{a_k}{2} e^{j2\pi (f_c + f_k) t} + \frac{a_k^*}{2} e^{j2\pi (f_c - f_k) t} + \frac{a_k}{2} e^{-j2\pi (f_c - f_k) t} + \frac{a_k^*}{2} e^{-j2\pi (f_c + f_k) t}$$

if the spectrum of $v(t)$ looks like:



hard to Transmit

then the spectrum of $x(t) = v(t) \cdot \text{carrier}(t)$ looks like:

easy to transmit