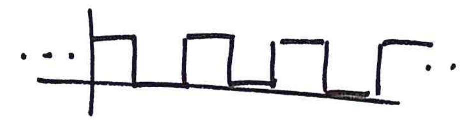
Lecture 7: The Fourier Series

Note: Every periodic signal can be represented as a sum of harmonic sinusoids. 

If $\quad x(t) = x(t + T_0)$ for all t

then $\quad x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos(2\pi k f_0 t + \phi_k)$

where $f_0 = 1/T_0$ = fundamental frequency

for some values of A_k's and ϕ_k's.

with complex exponentials, we can write the signal

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k f_0 t}$$

$$a_k = \begin{cases} A_0 & k=0 \\ \frac{A_k}{2} e^{j\phi_k} & k>0 \\ \frac{A_k}{2} e^{-j\phi_k} & k<0 \end{cases}$$

= Standard form of Fourier series.

Jean Baptiste Joseph Fourier

Born: 21 March 1768 in Auxerre, Bourgogne, France
Died: 16 May 1830 in Paris, France

Click the picture above
to see five larger pictures

Show birthplace location

Main Index Biographies index

Enter word or phrase | Search the MacTutor archive

Version for printing

Joseph Fourier's father was a tailor in Auxerre. After the death of his first wife, with whom he had three children, he remarried and Joseph was the ninth of the twelve children of this second marriage. Joseph's mother died went he was nine years old and his father died the following year.

His first schooling was at Pallais's school, run by the music master from the cathedral. There Joseph studied Latin and French and showed great promise. He proceeded in 1780 to the École Royale Militaire of Auxerre where at first he showed talents for literature but very soon, by the age of thirteen, mathematics became his real interest. By the age of 14 he had completed a study of the six volumes of Bézout's *Cours de mathématiques*. In 1783 he received the first prize for his study of Bossut's *Mécanique en général* T.

In 1787 Fourier decided to train for the priesthood and entered the Benedictine abbey of St Benoit-sur-Loire. His interest in mathematics continued, however, and he corresponded with C L Bonard, the professor of mathematics at Auxerre. Fourier was unsure if he was making the right decision in training for the priesthood. He submitted a paper on algebra to Montucla in Paris and his letters to Bonard suggest that he really wanted to make a major impact in mathematics. In one letter Fourier wrote

> *Yesterday was my 21st birthday, at that age Newton and Pascal had already acquired many claims to immortality.*

Fourier did not take his religious vows. Having left St Benoit in 1789, he visited Paris and read a paper on algebraic equations at the Académie Royale des Sciences. In 1790 he became a teacher at the Benedictine college, École Royale Militaire of Auxerre, where he had studied. Up until this time there had been a conflict inside Fourier about whether he should follow a religious life or one of mathematical research. However in 1793 a third element was added to this conflict when he became involved in politics and joined the local Revolutionary Committee. As he wrote:-

> *As the natural ideas of equality developed it was possible to conceive the sublime hope of establishing among us a free government exempt from kings and priests, and to free from this double yoke the long-usurped soil of Europe. I readily became enamoured of this cause, in my opinion the greatest and most beautiful which any nation has ever undertaken.*

Certainly Fourier was unhappy about the Terror which resulted from the French Revolution and he attempted to resign from the committee. However this proved impossible and Fourier was now firmly entangled with the Revolution and unable to withdraw. The revolution was a complicated affair with many factions, with broadly similar aims, violently opposed to each other. Fourier defended members of one faction while in Orléans. A letter describing events relates:-

> *Citizen Fourier, a young man full of intelligence, eloquence and zeal, was sent to Loiret. ... It seems that Fourier ... got up on certain popular platforms. He can talk very well and if he put forward the views of the Society of Auxerre he has done nothing blameworthy...*

This incident was to have serious consequences but after it Fourier returned to Auxerre and continued to work on the revolutionary committee and continued to teach at the College. In July 1794 he was arrested, the charges relating to the Orléans incident, and he was imprisoned. Fourier feared the he would go to the guillotine but, after Robespierre himself went to the guillotine, political changes resulted in Fourier being freed.

Later in 1794 Fourier was nominated to study at the École Normale in Paris. This institution had been set up for training teachers and it was intended to serve as a model for other teacher-training schools. The school opened in January 1795 and Fourier was certainly the most able of the pupils whose abilities ranged widely. He was taught by Lagrange, who Fourier described as

> *the first among European men of science,*

and also by Laplace, who Fourier rated less highly, and by Monge who Fourier described as

> *having a loud voice and is active, ingenious and very learned.*

Fourier began teaching at the Collège de France and, having excellent relations with Lagrange, Laplace and Monge, began further mathematical research. He was appointed to a position at the École Centrale des Travaux Publics, the school being under the direction of Lazare Carnot and Gaspard Monge, which was soon to be renamed École Polytechnique. However, repercussions of his earlier arrest remained and he was arrested again and imprisoned. His release has been put down to a variety of different causes, pleas by his pupils, pleas by Lagrange, Laplace or Monge or a change in the political climate. In fact all three may have played a part.

By 1 September 1795 Fourier was back teaching at the École Polytechnique. In 1797 he succeeded Lagrange in being appointed to the chair of analysis and mechanics. He was renowned as an outstanding lecturer but he does not appear to have undertaken original research during this time.

In 1798 Fourier joined Napoleon's army in its invasion of Egypt as scientific adviser. Monge and Malus were also part of the expeditionary force. The expedition was at first a great success. Malta was occupied on 10 June 1798, Alexandria taken by storm on 1 July, and the delta of the Nile quickly taken. However, on 1 August 1798 the French fleet was completely destroyed by Nelson's fleet in the Battle of the Nile, so that Napoleon found himself confined to the land that he was occupying. Fourier acted as an administrator as French type political institutions and administration was set up. In particular he helped establish educational facilities in Egypt and carried out archaeological explorations.

While in Cairo Fourier helped found the Cairo Institute and was one of the twelve members of the mathematics division, the others included Monge, Malus and Napoleon himself. Fourier was elected secretary to the Institute, a position he continued to hold during the entire French occupation of Egypt. Fourier was also put in charge of collating the scientific and literary discoveries made during the time in Egypt.

Napoleon abandoned his army and returned to Paris in 1799, he soon held absolute power in France. Fourier returned to France in 1801 with the remains of the expeditionary force and resumed his post as Professor of Analysis at the École Polytechnique. However Napoleon had other ideas about how Fourier might serve him and wrote:-

> *... the Prefect of the Department of Isère having recently died, I would like to express my confidence in citizen Fourier by appointing him to this place.*

Fourier was not happy at the prospect of leaving the academic world and Paris but could not refuse Napoleon's request. He went to Grenoble where his duties as Prefect were many and varied. His two greatest achievements in this administrative position were overseeing the operation to drain the swamps of Bourgoin and supervising the construction of a new highway from Grenoble to Turin. He also spent much time working on the *Description of Egypt* which was not completed until 1810 when Napoleon made changes, rewriting history in places, to it before publication. By the time a second edition appeared every reference to Napoleon would have been removed.

It was during his time in Grenoble that Fourier did his important mathematical work on the theory of heat. His work on the topic began around 1804 and by 1807 he had completed his important memoir *On the Propagation of Heat in Solid Bodies*. The memoir was read to the Paris Institute on 21 December 1807 and a committee consisting of Lagrange, Laplace, Monge and Lacroix was set up to report on the work. Now this memoir is very highly regarded but at the time it caused controversy.

There were two reasons for the committee to feel unhappy with the work. The first objection, made by Lagrange and Laplace in 1808, was to Fourier's expansions of functions as trigonometrical series, what we now call Fourier series. Further clarification by Fourier still failed to convince them. As is pointed out in [4]:-

> *All these are written with such exemplary clarity - from a logical as opposed to calligraphic point of view - that their inability to persuade Laplace and Lagrange ... provides a good index of the originality of Fourier's views.*

The second objection was made by Biot against Fourier's derivation of the equations of transfer of heat. Fourier had not made reference to Biot's 1804 paper on this topic but Biot's paper is certainly incorrect. Laplace, and later Poisson, had similar objections.

The Institute set as a prize competition subject the propagation of heat in solid bodies for the 1811 mathematics prize. Fourier submitted his 1807 memoir together with additional work on the cooling of infinite solids and terrestrial and radiant heat. Only one other entry was received and the committee set up to decide on the award of the prize, Lagrange, Laplace, Malus, Haüy and Legendre, awarded Fourier the prize. The report was not however completely favourable and states:-

> *... the manner in which the author arrives at these equations is not exempt of difficulties and that his analysis to integrate them still leaves something to be desired on the score of generality and even rigour.*

With this rather mixed report there was no move in Paris to publish Fourier's work.

When Napoleon was defeated and on his way to exile in Elba, his route should have been through Grenoble. Fourier managed to avoid this difficult confrontation by sending word that it would be dangerous for Napoleon. When he learnt of Napoleon's escape from Elba and that he was marching towards Grenoble with an army, Fourier was extremely worried. He tried to persuade the people of Grenoble to oppose Napoleon and give their allegiance to the King. However as Napoleon marched into the town by one gate Fourier left in haste by another.

Napoleon was angry with Fourier who he had hoped would welcome his return. Fourier was able to talk his way into favour with both sides and Napoleon made him Prefect of the Rhône. However Fourier soon resigned on receiving orders, possibly from Carnot, that the was to remove all administrators with royalist sympathies. He could not have completely fallen out with Napoleon and Carnot, however, for on 10 June 1815, Napoleon awarded him a pension of 6000 francs, payable from 1 July. However Napoleon was defeated on 1 July and Fourier did not receive any money. He returned to Paris.

Fourier was elected to the Académie des Sciences in 1817. In 1822 Delambre, who was the Secretary to the mathematical section of the Académie des Sciences, died and Fourier together with Biot and Arago applied for the post. After Arago withdrew the election gave Fourier an easy win. Shortly after Fourier became Secretary, the Académie published his prize winning essay *Théorie analytique de la chaleur* in 1822. This was not a piece of political manoeuvring by Fourier however since Delambre had arranged for the printing before he died.

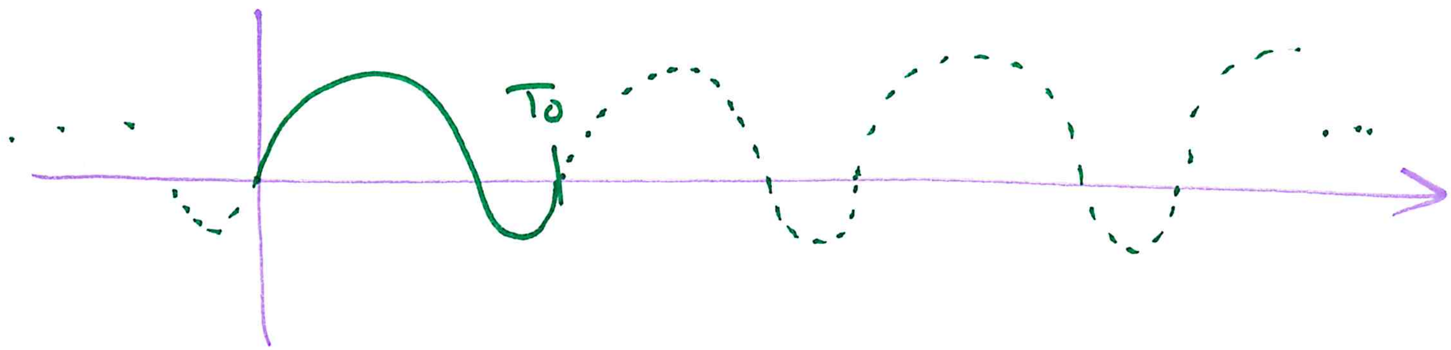
During Fourier's eight last years in Paris he resumed his mathematical researches and published a number of papers, some in pure mathematics while some were on applied mathematical topics. His life was not without problems however since his theory of heat still provoked controversy. Biot claimed priority over Fourier, a claim which Fourier had little difficulty showing to be false. Poisson, however, attacked both Fourier's mathematical techniques and also claimed to have an alternative theory. Fourier wrote *Historical Précis* Ⓣ as a reply to these claims but, although the work was shown to various mathematicians, it was never published.

Fourier's views on the claims of Biot and Poisson are given in the following, see [4]:-

> *Having contested the various results* [*Biot and Poisson*] *now recognise that they are exact but they protest that they have invented another method of expounding them and that this method is excellent and the true one. If they had illuminated this branch of physics by important and general views and had greatly perfected the analysis of partial differential equations, if they had established a principal element of the theory of heat by fine experiments ... they would have the right to judge my work and to correct it. I would submit with much pleasure .. But one does not extend the bounds of science by presenting, in a form said to be different, results which one has not found oneself and, above all, by forestalling*

How do we find the Fourier series coefficients (a_k's) for an arbitrary periodic signal $x(t)$ (with period T_0)?

Recall: finite duration signals can be thought of as one period of a periodic signal.



Fourier Analysis (finding Fourier coefficients)

Given a periodic signal $x(t)$, with period T_0, find its Fourier series representation

Compute

$$a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j2\pi k f_0 t} \, dt$$

where $f_0 = 1/T_0$ = "fundamental frequency"

Verification:

Recall $$x(t) = \sum_{\cancel{k}\,\ell=-\infty}^{\infty} a_{\cancel{k}\,\ell}\, e^{+j2\pi f_0 \cancel{k}\,\ell t}$$

Goal: show $b_k = a_k$

$$b_k = \frac{1}{T_0}\int_0^{T_0} x(t)\, e^{-j2\pi f_0 k t}\, dt$$

$$= \frac{1}{T_0}\int_0^{T_0} \left[\sum_{\ell=-\infty}^{\infty} a_\ell\, e^{j2\pi f_0 \ell t}\right] e^{-j2\pi f_0 k t}\, dt$$

$$= \frac{1}{T_0}\sum_{\ell=-\infty}^{\infty} a_\ell \underbrace{\left(\int_0^{T_0} e^{j2\pi f_0 \ell t - j2\pi f_0 k t}\, dt\right)}_{(*)\ \text{focus here}}$$

$$\int_b e^{ax}\,dx = \frac{1}{a} e^{ax} \Big|_{x=b}$$

5

if $k = l$

then $$ⓧ = \int_0^{T_0} e^{j2\pi \cancel{f_0 t} (l-k)^{\,0}}\,dt = \int_0^{T_0} 1\,dt = T_0$$

if $k \neq l$

then $$ⓧ = \int_0^{T_0} e^{j2\pi f_0 t (l-k)}\,dt = \frac{1}{j2\pi f_0 (l-k)} \left[e^{j2\pi f_0 t (l-k)} \right] \Big|_0^{T_0}$$

$$= \frac{1}{j2\pi f_0 (l-k)} \left[e^{j2\pi \cancel{f_0 T_0}^{\,1} (l-k)} - e^{0} \right]$$

$$= \frac{1}{j2\pi f_0 (l-k)} (1-1) = 0$$

remember
$f_0 = 1/T_0$

$e^{j2\pi k} = 1$ if k is integer

just showed

$$\circledast = \int_0^{T_0} e^{j2\pi f_0 t\,(l-k)}\,dt = \begin{cases} T_0 & \text{if } k = l \\ 0 & \text{if } k \neq l \end{cases}$$

$$b_k = \frac{1}{T_0}\sum_{l=-\infty}^{\infty} a_l \cdot \circledast = \frac{1}{T_0}\cdot a_k \cdot T_0 = a_k \quad \checkmark$$

Ex: $x(t) = \cos(50\pi t)$. what are fourier coeffs?

$f_0 = 25$ Hz $\Rightarrow T_0 = \frac{1}{25}$ s.

by Euler's identity

$$x(t) = \frac{1}{2} e^{j2\pi\cdot 25t} + \frac{1}{2} e^{-j2\pi\cdot 25t}$$

$j2\pi 25 t = j2\pi f_0 \cdot 1 \cdot t$
$\Rightarrow k=1$

$k=-1$

$$a_k = \begin{cases} 1/2 & \text{if } k = 1, -1 \\ 0 & \text{otherwise} \end{cases}$$

$$x(t) = \cos(2\pi(25)t) \qquad f_0 = 25, \quad T_0 = \frac{1}{25}$$

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)\, e^{-j2\pi f_0 k t}\, dt$$

$$= \frac{1}{T_0}\int_0^{T_0} \left[\frac{1}{2}e^{j2\pi f_0 t} + \frac{1}{2}e^{-j2\pi f_0 t}\right] e^{-j2\pi f_0 k t}\, dt$$

$$= \underbrace{\frac{1}{T_0}\int_0^{T_0}\left(\frac{1}{2}e^{j2\pi f_0 t(1-k)}\right.}_{=\begin{cases}\frac{1}{2} & \text{if } k=1\\ 0 & \text{o.w.}\end{cases}} \underbrace{\left. + \frac{1}{2}e^{j2\pi f_0 t(-1-k)}\right) dt}_{=\begin{cases}\frac{1}{2} & \text{if } k=-1\\ 0 & \text{o.w.}\end{cases}}$$

$$= \begin{cases} \frac{1}{2} & \text{if } k = 1, -1 \\ 0 & \text{otherwise} \end{cases}$$