Lecture 0

The Fourier Series

If $x(t)$ is periodic with period T_0, then

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k f_0 t} \qquad \text{where } f_0 = 1/T_0$$

and

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t) e^{-j2\pi k f_0 t} dt$$

Equivalent but less compact / ~~less~~ non-standard

$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos(2\pi f_0 k t + \phi_k)$$

where

$$a_k = \begin{cases} \frac{A_k}{2} e^{j\phi_k} & \text{for } k > 0 \\ A_0 & \text{for } k = 0 \\ \frac{A_k}{2} e^{-j\phi_k} = a_k^* & \text{for } k < 0 \end{cases}$$

Ex. 1 $x(t) = \cos(50\pi t)$ $f_0 = 25\text{ Hz} \Rightarrow T_0 = 1/25\text{ s.}$

Method A:

use Euler's id: $x(t) = \frac{1}{2} e^{j2\pi \cdot 25t} + \frac{1}{2} e^{-j2\pi \cdot 25 t}$ $\Leftarrow$

want $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k f_0 t}$

= if k=1

if k=-1

$= a_1 e^{j2\pi \cdot 1 f_0 t} + a_{-1} e^{j2\pi(-1) f_0 t}$ $\Leftarrow$

$= \frac{1}{2} e^{j50\pi t} + \frac{1}{2} e^{-j50\pi t}$

$a_1 = 1/2 \qquad a_{-1} = 1/2$

$a_k = 0$ for $k \neq 1, -1$

Method B :

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)\, e^{-j2\pi f_0 k t}\, dt$$

$$= \frac{1}{T_0}\int_0^{T_0}\left[\frac{1}{2} e^{j2\pi f_0 t} + \frac{1}{2} e^{-j2\pi f_0 t}\right] e^{-j2\pi f_0 k t}\, dt$$

$$= \frac{1}{T_0}\int_0^{T_0}\left[\frac{1}{2} e^{j2\pi f_0 t(1-k)} + \frac{1}{2} e^{j2\pi f_0 t(-1-k)}\right] dt$$

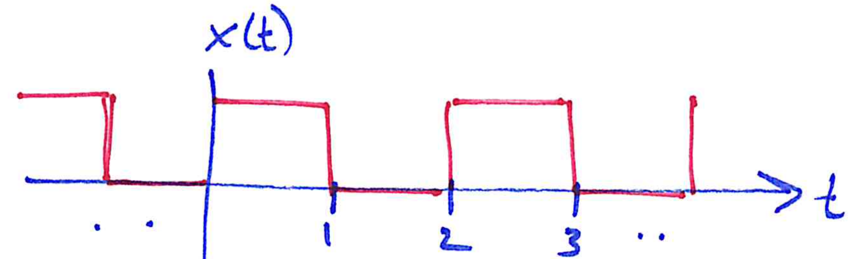
$$= \frac{1}{2T_0}\left[\underbrace{\int_0^{T_0} e^{j2\pi f_0 t(1-k)}\, dt}_{=\begin{cases} T_0 & \text{if } k=1 \\ 0 & \text{o.w.}\end{cases}} + \underbrace{\int_0^{T_0} e^{j2\pi f_0 t(-1-k)}\, dt}_{=\begin{cases} T_0 & \text{if } k=-1 \\ 0 & \text{o.w.}\end{cases}}\right]$$

$$\boxed{\int_0^{T_0} e^{j2\pi t(k-l)f_0}\, dt = \begin{cases} T_0 & k=l \\ 0 & k\neq l\end{cases}}$$

$$= \begin{cases} 1/2 & k=1 \\ 1/2 & k=-1 \\ 0 & k \text{ other}\end{cases}$$

Ex 2: Square wave

$$x(t) = \begin{cases} 1 & 0 < t \leq 1 \\ 0 & 1 < t \leq 2 \end{cases}$$



$T_0 = 2\text{ s}$

$f_0 = \frac{1}{2}\text{ Hz}$

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)\, e^{-j2\pi f_0 k t}\, dt$$

$$= \frac{1}{2}\int_0^{1} 1\cdot e^{-j2\pi f_0 k t}\, dt + \underbrace{\frac{1}{2}\int_1^{2} 0\cdot e^{-j2\pi f_0 k t}\, dt}_{=0}$$

for $k \neq 0$

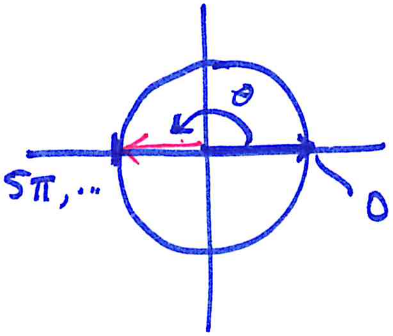
$$= \frac{1}{2}\left.\frac{e^{-j2\pi f_0 k t}}{-j\cancel{2}\pi \cancel{f_0} k}\right|_0^1$$

$$\int e^{at}\,dt = \frac{e^{at}}{a}$$

$$= \frac{1}{-j2\pi k}\left[e^{-j\cancel{2}\pi\cancel{f_0}k} - 1\right] = \begin{cases} 0 & \text{if } k \text{ even}, k\neq 0 \\ 1/j\pi k & \text{if } k \text{ odd}, k \neq 0 \\ 1/2 & \text{if } k=0 \end{cases}$$

$$e^{-j\pi k} = \begin{cases} 1 & k \text{ even} \\ -1 & k \text{ odd} \end{cases}$$

$\theta = \pi, 3\pi, 5\pi, \ldots$



$0, 2\pi, 4\pi, 6\pi, \ldots$

$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

$$a_0 = \frac{1}{2}\int_0^1 1 \cdot e^0 \, dt = \frac{1}{2}\int_0^1 1 \, dt = 1/2$$

for k odd

$$a_k = \frac{-1}{j2\pi k}\left[e^{-j\cancel{2}\pi(\cancel{1/2})k} - 1\right]$$

$$= \frac{-1}{j2\pi k}\left[e^{-j\pi k} - 1\right] = \frac{-1}{j2\pi k}\left[-1-1\right] = \frac{-1}{j2\pi k}(-2) = \frac{1}{j\pi k}$$

$$e^{-jk\pi} = \cos(-k\pi) + j\sin(-k\pi)$$

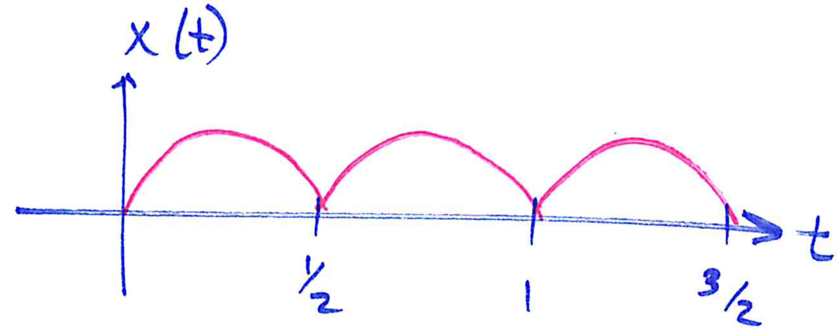
$\cos(k\pi) = -1$ if k odd

Ex 2:

$$x(t) = |\sin(2\pi t)|$$

$T_0 = \frac{1}{2}$ s

$f_0 = 2$ Hz



$$\sin\theta = \frac{1}{2j}\left(e^{j\theta} - e^{-j\theta}\right)$$

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)\, e^{-j2\pi k f_0 t}\, dt$$

$$= \frac{1}{1/2}\int_0^{1/2} |\sin(2\pi t)|\, e^{-j2\pi k \cdot 2t}\, dt = 2\int_0^{1/2} \underline{\sin(2\pi t)}\, e^{-j2\pi k \cdot 2t}\, dt$$

$$= \cancel{2}\int_0^{1/2} \frac{1}{\cancel{2}j}\left[e^{j2\pi t} - e^{-j2\pi t}\right] e^{-j2\pi k \cdot 2t}\, dt$$

$$= \int_0^{1/2}\left[\frac{1}{j}\, e^{j2\pi t(1-2k)} - \frac{1}{j}\, e^{-j2\pi t(1+2k)}\right] dt$$