Lecture 11: Spectra of Sampled Signals

recall digital frequency $\hat{f} = \dfrac{f_0}{f_s}$

note that if $f_s > 2 f_{max}$, then $\hat{f}_{max} = \dfrac{f_{max}}{f_s} < \frac{1}{2}$ cycles/sample

Recall the Fourier series expansion/ representation /synthesis:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j 2\pi f_0 k t}$$

If $x(t)$ only has low frequencies

$$x(t) = \sum_{k=-\underline{N}}^{\underline{N}} a_k e^{j 2\pi f_0 k t}$$

$\Rightarrow f_{max} = N f_0$

$\Rightarrow$ to avoid aliasing, sample @ freq $f_s > 2N f_0$

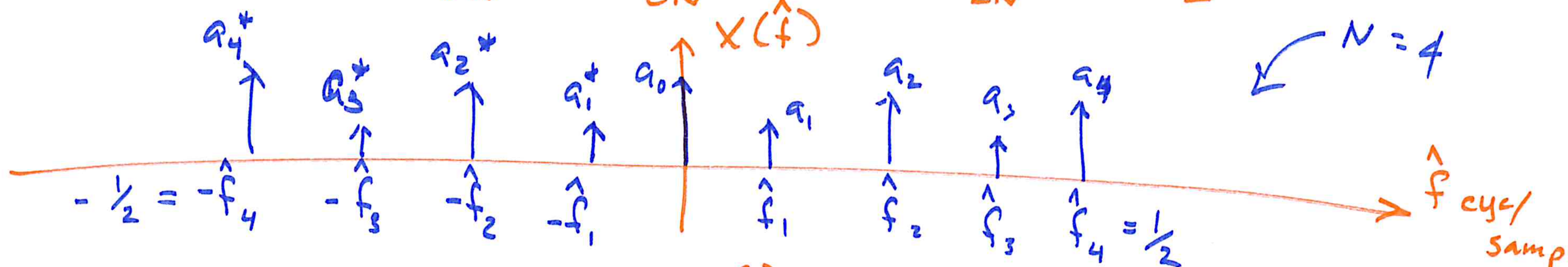
Suppose we sample $x(t)$ w/ freq $f_s = 2Nf_0$ $\left(T_s = 1/f_s\right)$ ②

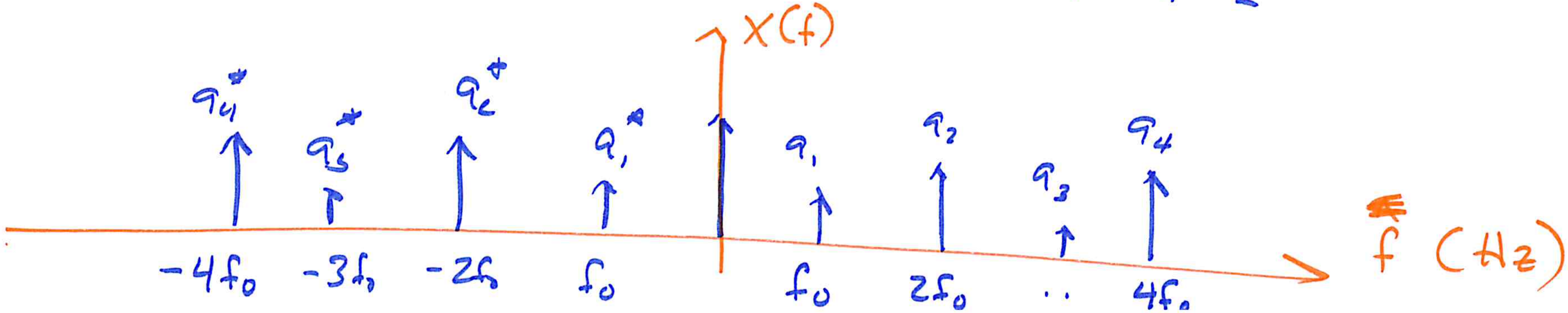
$$x[n] = x(nT_s) = \sum_{k=-N}^{N} a_k e^{j2\pi k \frac{f_0}{f_s} n}$$

$$= \sum_{k=-N}^{N} a_k e^{j2\pi \underbrace{\frac{k}{2N}}_{\hat{f}_k} n}$$

$\Rightarrow$ digital frequencies in $x[n]$ are

$$0,\ \pm\frac{1}{2N},\ \pm\frac{2}{2N},\ \pm\frac{3}{2N},\ \ldots,\ \pm\frac{N-1}{2N},\ \pm\frac{1}{2}$$





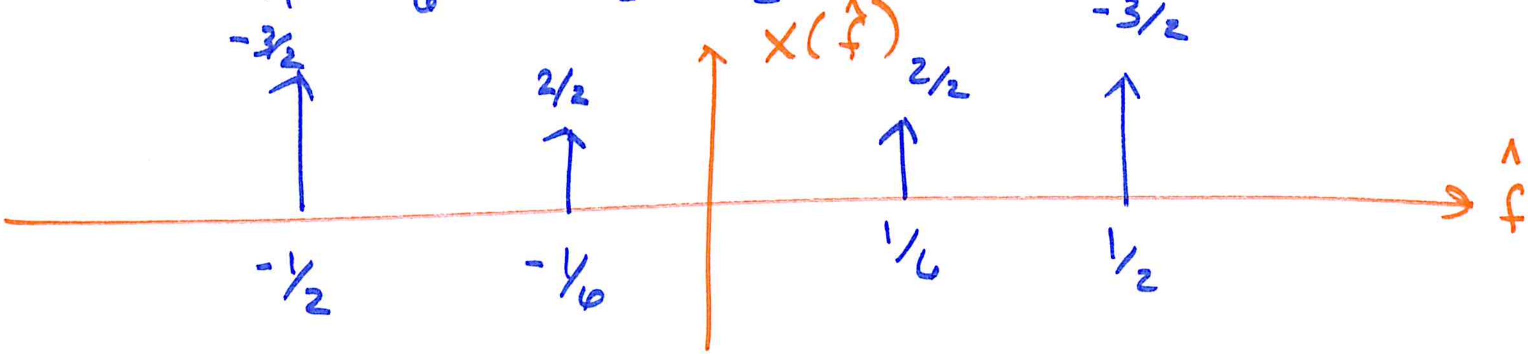
Ex: $x(t) = 2\cos(2\pi 100t) - 3\cos(2\pi 300t)$

$f_s = 600$ sample/sec $\qquad nT_s = n/f_s = n/600$

$$x[n] = 2\cos\left(2\pi \frac{100}{600} n\right) - 3\cos\left(2\pi \frac{300}{600} n\right)$$

$A\cos(\theta) = \frac{A}{2}e^{j\theta} + \frac{A}{2}e^{j}$

$$\hat{f}_1 = \frac{1}{6}, \quad \hat{f}_2 = \frac{1}{2}$$



$\Rightarrow$ same shape as original spectrum

$f_s = 350$ samples/sec.

$$x[n] = 2\cos\left(2\pi \cancel{\frac{100}{350}}^{\,2/7} n\right) - 3\cos\left(2\pi \cancel{\frac{300}{350}}^{\,6/7} n\right)$$

$$= 2\cos\left(2\pi \frac{2}{7} n\right) - 3\cos\left(2\pi \frac{1}{7} n\right)$$

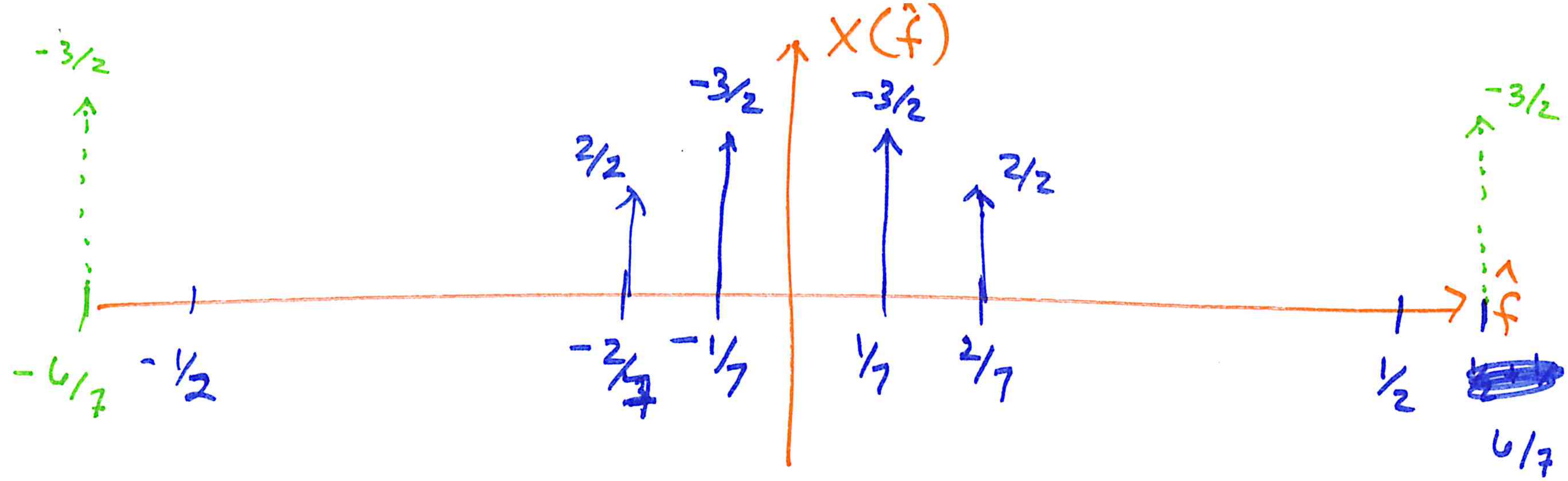
$$\hat{f}_1 = 2/7, \quad \hat{f}_2 = 1/7$$

$\cos(-2\pi \frac{6}{7} n)$

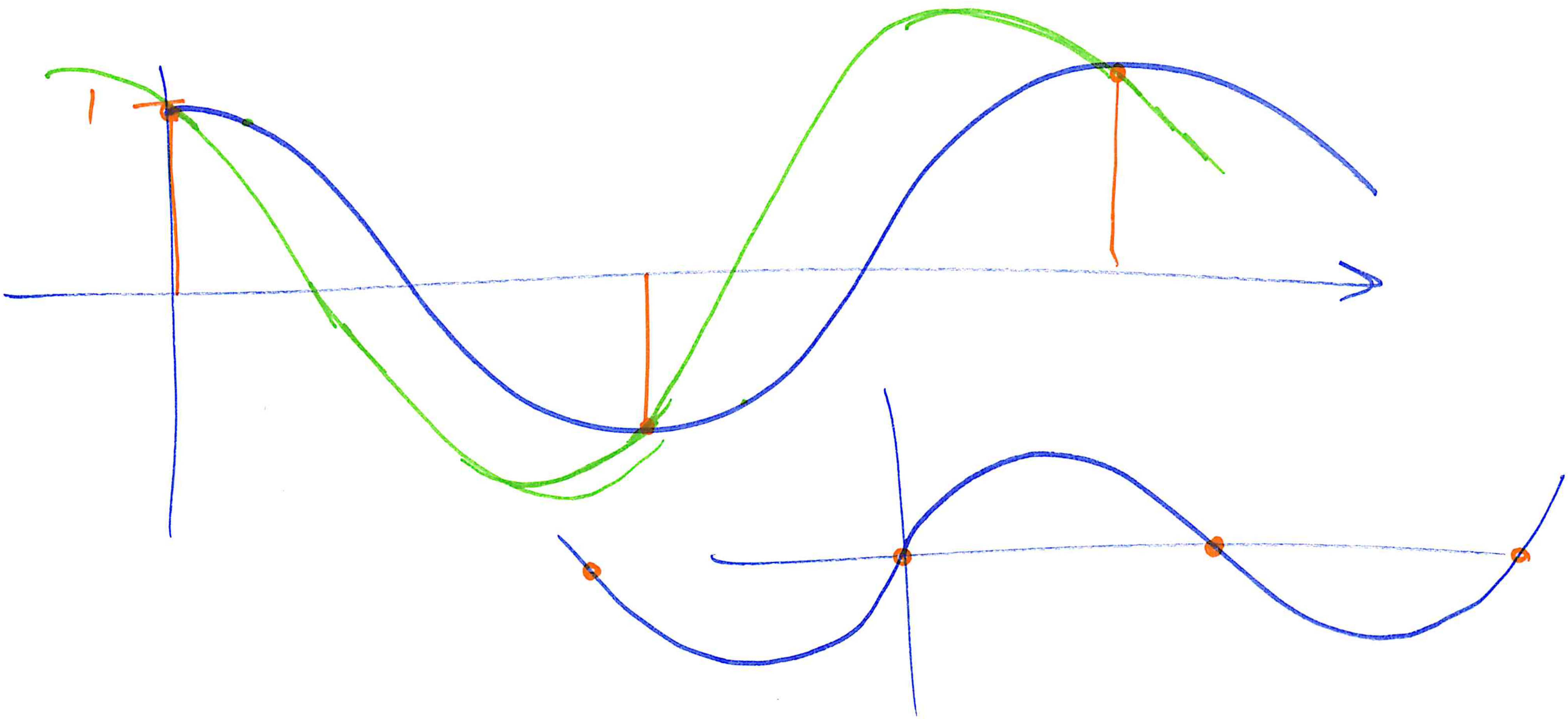
$= \cos(2\pi n - 2\pi \frac{6}{7} n)$

$= \cos(2\pi (1 - 6/7) n)$

$= \cos(2\pi \frac{1}{7} n)$



X(f̂)
-3/2
-3/2
-3/2
-3/2
2/2
2/2
-6/7
-1/2
-2/7
-1/7
1/7
2/7
1/2
6/7
f̂



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Ultimately we want to reconstruct a signal from samples. Our strategy: assume $x(t)$ is low-frequency and find the lowest-frequency sum of sinusoids that matches the samples. Aliasing makes that process inaccurate.