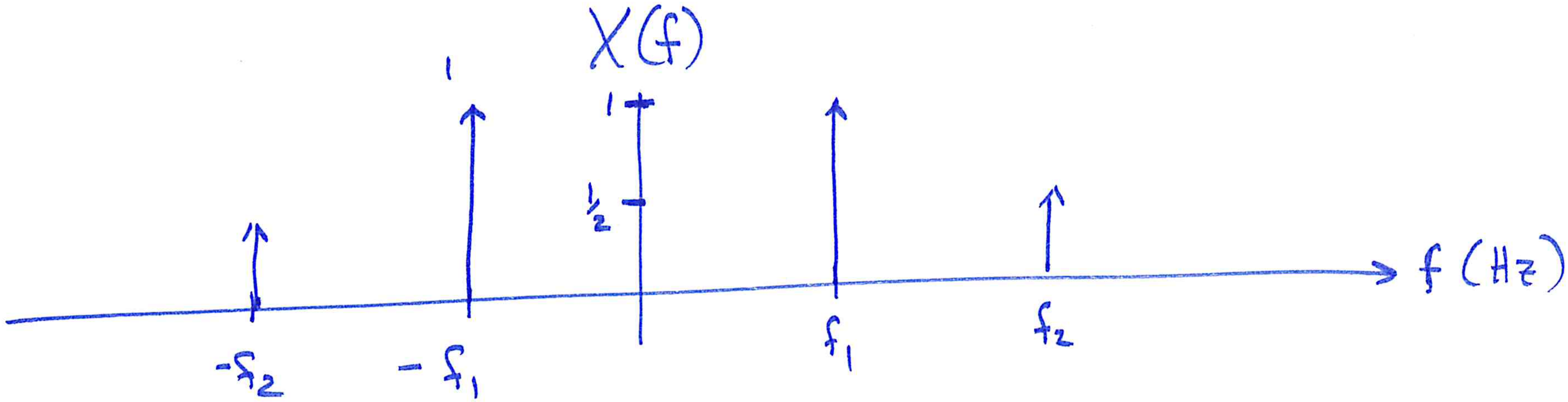
Lec. 12 – Spectra of Sampled Signals and the Discrete Fourier Transform

consider $x(t) = 2\cos(2\pi f_1 t) + \cos(2\pi f_2 t)$

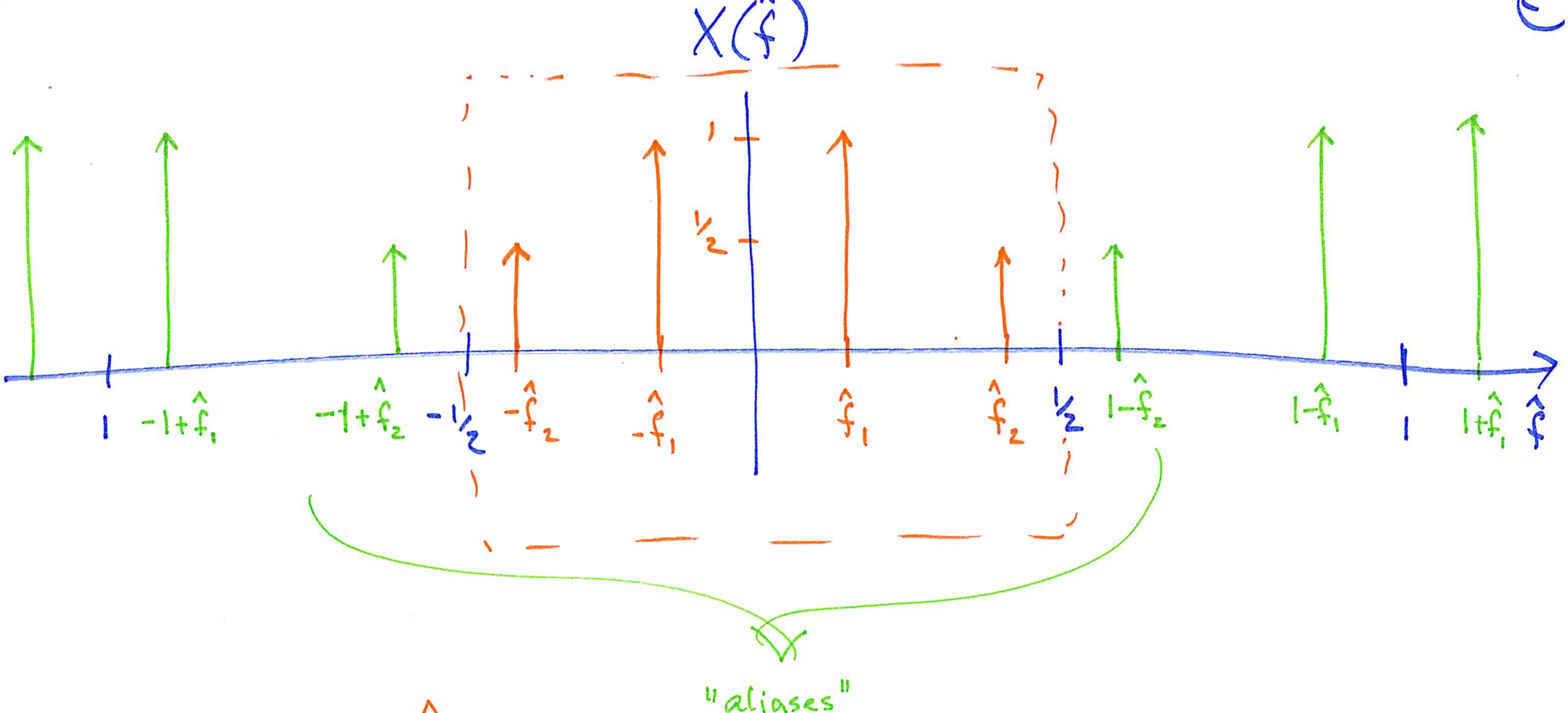


sample @ $f_s > 2 f_{max}$; $\hat{f}_1 = \frac{f_1}{f_s}$, $\hat{f}_2 = f_2/f_s$

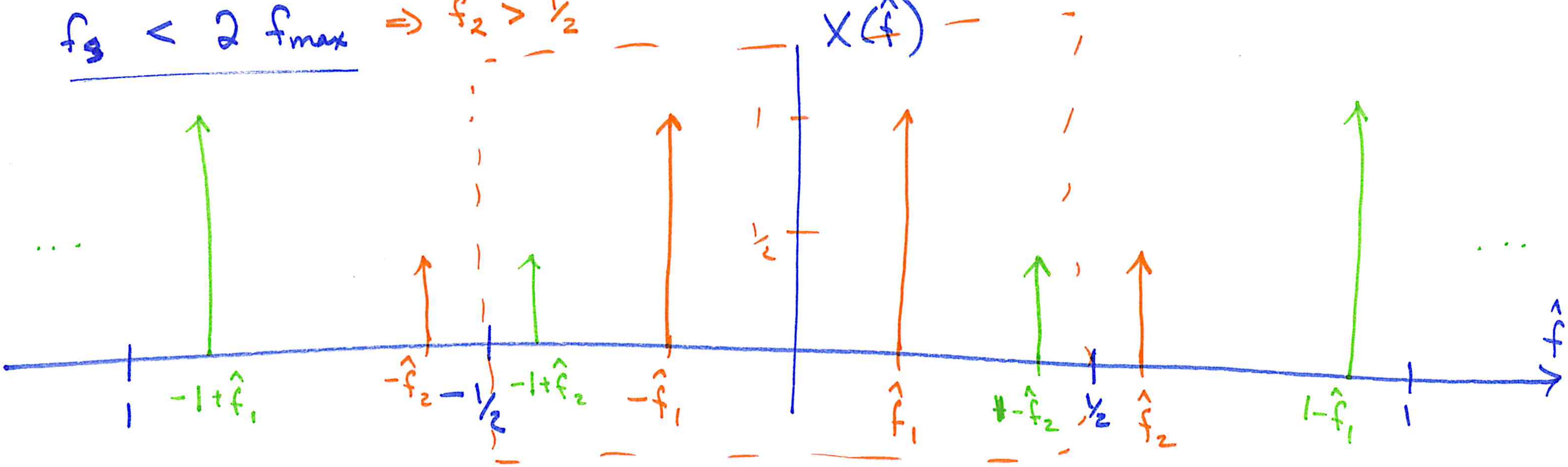
$$x[n] = x(nT_s) = 2\cos\left(2\pi \frac{f_1}{f_s} n\right) + \cos\left(2\pi \frac{f_2}{f_s} n\right) = 2\cos(2\pi \hat{f}_1 n) + \cos(2\pi \hat{f}_2 n)$$

$$\equiv 2\cos(\pm 2\pi(\hat{f}_1 + k_1)n) + \cos(\pm 2\pi(\hat{f}_2 + k_2)n)$$

for any integers k_1, k_2



$f_s < 2 f_{max} \Rightarrow \hat{f}_2 > 1/2$



The Discrete Fourier Transform (DFT)

One use: determine the spectrum of an arbitrary sampled signal.

To get the spectrum of an analog/continuous time signal,
use Fourier series

To get the spectrum of a digital/discrete time signal,
use the DFT

Integral to JPEG, MP3, MP4, WiFi, GPS,

DFT: mathematical operation or representation

Fast Fourier Transform (FFT) - fast computer algorithm to compute the DFT

Recall Fourier series:

A periodic signal $x(t)$ w/ period T_0 (or finite duration signal) could be represented exactly by FS coefficients:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k f_0 t}, \quad f_0 = 1/T_0 \qquad \text{(synthesis)}$$

where

$$a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j2\pi k f_0 t} dt \qquad \text{(analysis)}$$

DFT: same idea but ~~but~~ for digital / discrete time signals.

The DFT

Any discrete time signal $x[n]$, $n = 0, \dots, N-1$ (Length N)
can be represented exactly by DFT coefficients:

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j2\pi kn/N} \qquad \text{for } n = 0, 1, \dots, N-1$$

where

$$\underbrace{X[k]}_{k^{th}\text{ DFT coefficient}} = \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N}, \qquad \text{for } k = 0, 1, \dots, N-1$$

N in DFT $\approx$ T_0 in FS