Lecture 13: The Discrete Fourier Transform (DFT)

The DFT is a discrete version of the Fourier Series:

Fourier Series

Any periodic signal x(t) with period T_0 (or finite duration signal) can ~~represent~~ be represented it in exactly terms of its Fourier series

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{+j2\pi f_0 k t} \qquad \text{w/ } f_0 = 1/T_0$$

where the Fourier series coefficients are

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t) e^{-j2\pi \underline{f_0} k t}\, dt$$

fundamental freq $f_0 = 1/T_0$

harmonics $= 0, 1/T_0, 2/T_0, 3/T_0, \ldots$

Discrete Fourier Transform

Any discrete-time signal x[n] of length N can be represented exactly in terms of its DFT

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[\underline{k}] e^{+j2\pi \frac{kn}{N}}$$

where the DFT coefficients are

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi \frac{kn}{N}}$$

fundamental frequency $1/N$

harmonics $= 0, 1/N, 2/N, 3/N, \ldots$

Example: $x[n]$ = length-8 signal $(N=8)$

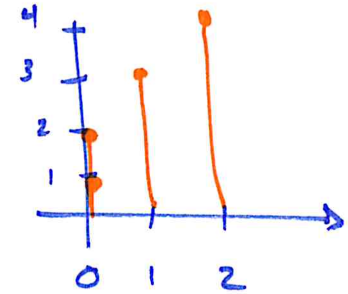
$$x[n] = \frac{1}{8}\sum_{k=0}^{7} X[k]\, e^{j2\pi nk/8}$$

$$= \frac{1}{8}X[0] + \frac{1}{8}X[1]e^{j2\pi n/8} + \frac{1}{8}X[2]e^{j4\pi n/8} +$$

$$\cdots + \frac{1}{8}X[7]e^{j14\pi n/8}$$

"building block" are $e^{j2\pi nk/8}$ complex sinusoids @ frequencies that are integer multiples of $1/8$ $(0, 1/8, 2/8, \ldots, 7/8)$

Example: $x[n] = [2, 3, 4]$ $\quad N = 3$



$$X[0] = 2 + 3 + 4 = 9$$

$$X[1] = 2 \cdot 1 + 3 \cdot e^{-j2\pi \, 1/3} + 4e^{-j2\pi \cdot 2/3} = -1.5 + 0.867j$$

$$X[2] = 2 \cdot 1 + 3e^{-j2\pi \, 2/3} + 4e^{-j2\pi \, 4/3} = -1.5 - 0.867j$$

DFT of $x = [9 \quad -1.5 + 0.867j \quad -1.5 - 0.867j]$

Another way to write the DFT:

Assume $x[n]$ is real-valued, N is odd

$$x[n] = A_0 + \sum_{k=1}^{\frac{N-1}{2}} A_k \cos\left(2\pi \frac{kn}{N} + \phi_k\right)$$

$$= A_0 + \sum_{k=1}^{\frac{N-1}{2}} \frac{A_k}{2} e^{j2\pi kn/N} e^{j\phi_k} + \frac{A_k}{2} e^{-j2\pi kn/N} e^{-j\phi_k}$$

let $a_k = \frac{A_k}{2} e^{j\phi_k}$

$$= A_0 + \sum_{k=1}^{\frac{N-1}{2}} a_k e^{j2\pi kn/N} + a_k^* e^{-j2\pi kn/N}$$

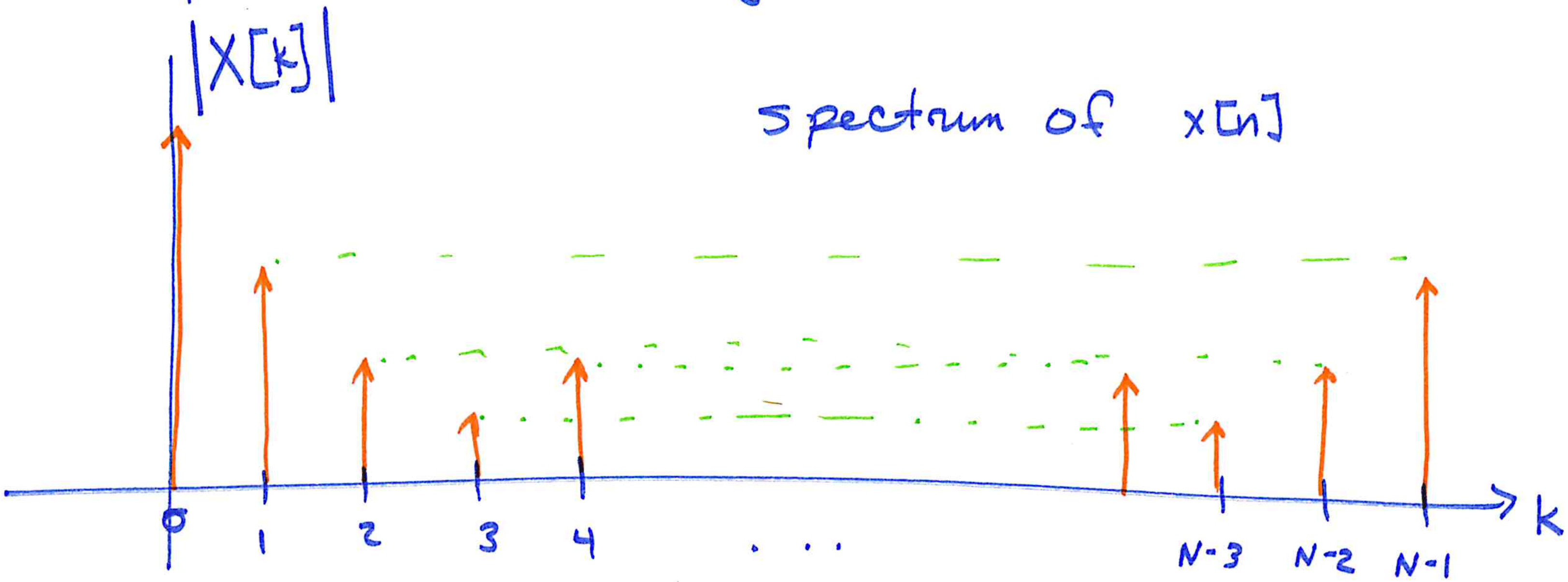
let $k' = N - k$

$$= A_0 + \sum_{k=1}^{\frac{N-1}{2}} a_k e^{j2\pi kn/N} + \sum_{k'=\frac{N+1}{2}}^{N-1} a_{N-k'}^* e^{j2\pi k'n/N}$$

$$= \sum_{k=0}^{N-1} \frac{1}{N} X[k] e^{j2\pi kn/N}$$

where
$$X[k] = \begin{cases} N A_0 & \text{if } k = 0 \\ N\frac{A_k}{2} e^{j\phi_k} & \text{if } k = 1, 2, \ldots, \frac{N-1}{2} \\ N\frac{A_{N-k}}{2} e^{-j\phi_{N-k}} & \text{if } k = \frac{N+1}{2}, \ldots, N-1 \end{cases}$$

$X[k]$ tells us the amplitude + phase of each sinusoidal component of our signal $x[n]$.

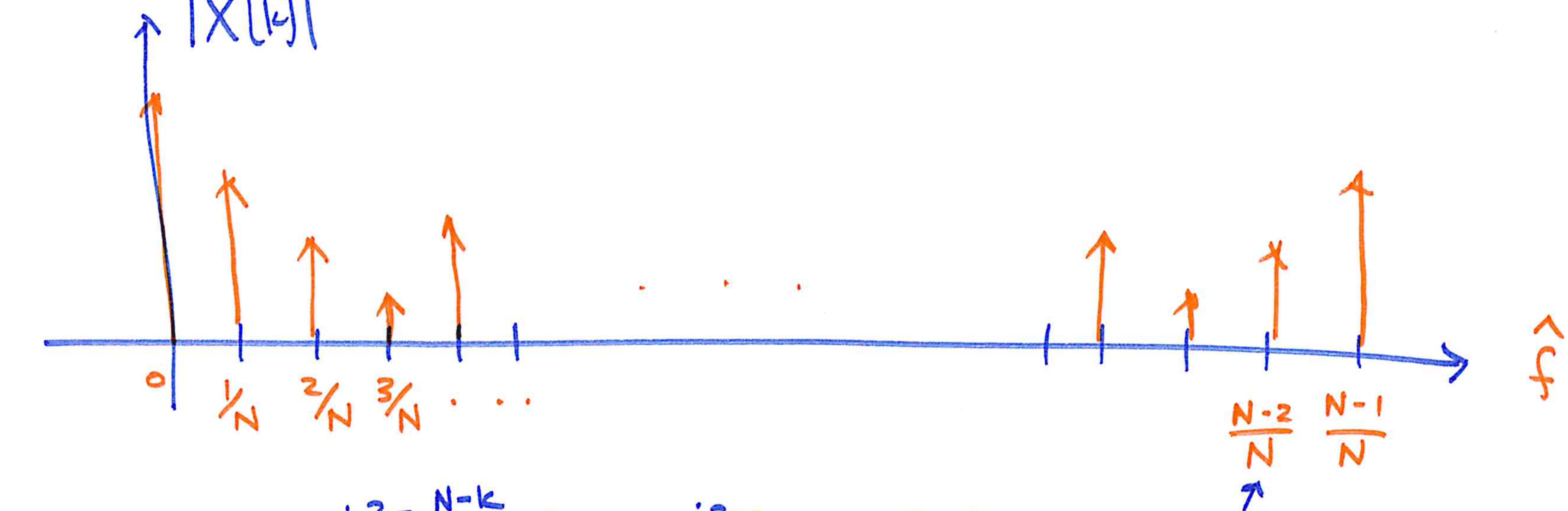


$X[k]$ = weight of the sinusoid $e^{j2\pi kn/N}$

= complex amplitude at frequency k/N cycles/sample

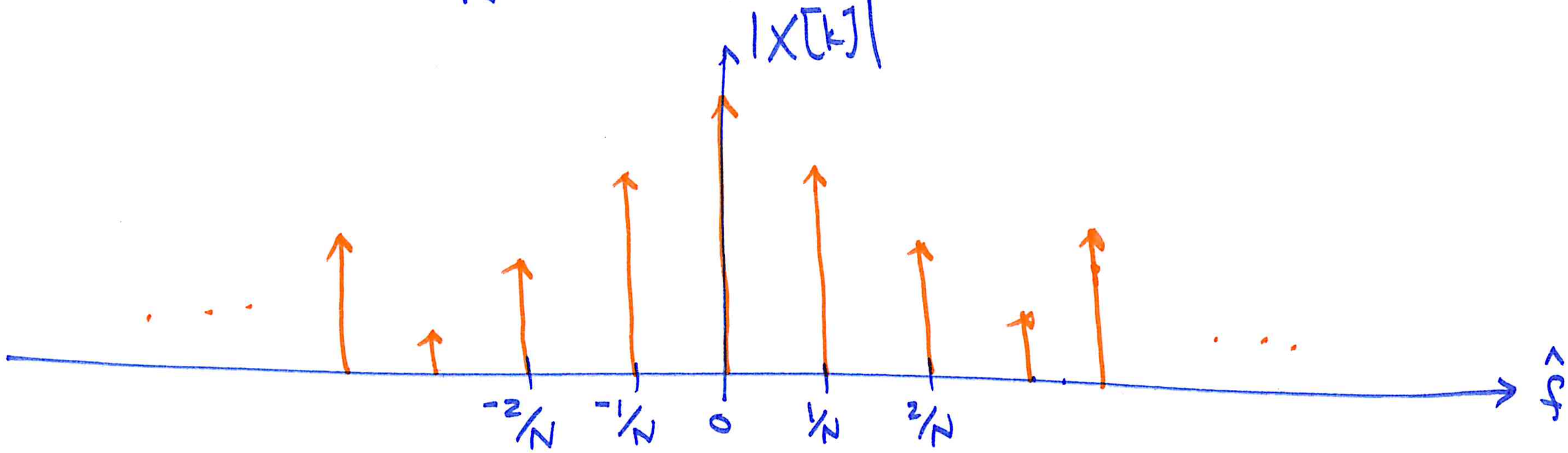
= amount of energy in signal at frequency k/N

⇒ plot $|X[k]|$ vs. digital frequency



notice $e^{j2\pi \frac{N-k}{N} \cdot n} = e^{j2\pi n} e^{-j2\pi kn/N} = e^{-j2\pi kn/N}$

so frequency $\frac{N-k}{N}$ is equivalent to $-k/N$



We can also relate the spectrum back to the continuous-time spectrum because $\hat{f} = f/f_s$