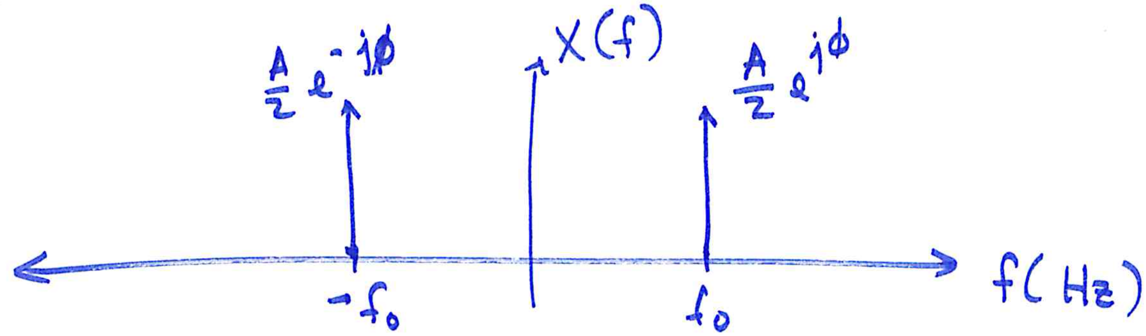
Lecture 14 DFT and spectra of sampled signals

Start w/ a continuous-time signal

$$x(t) = A\cos(2\pi f_0 t + \phi)$$

Spectrum



Now sample $x(t)$ w/ freq $f_s > 2f_0$ for T sec.

$$x[n] = x(nT_s) = A\cos(2\pi f_0 n T_s + \phi)$$

$$= A\cos(2\pi \hat{f} n + \phi)$$

for $n = 0, 1, \ldots, N-1$

where $\hat{f} = f_0 / f_s$

where $N = \lfloor T \cdot f_s \rfloor + 1$

The DFT:

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, e^{j2\pi kn/N}$$

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j2\pi kn/N}$$

for $x[n] = A\cos(2\pi \hat{f} n + \phi) = \frac{A}{2} e^{j\phi} e^{j2\pi \hat{f} n} + \frac{A}{2} e^{-j\phi} e^{-j2\pi \hat{f} n}$

$$X[k] = \sum_{n=0}^{N-1} \left(\frac{A}{2} e^{j\phi} e^{j2\pi \hat{f} n} + \frac{A}{2} e^{-j\phi} e^{-j2\pi \hat{f} n} \right) e^{-j2\pi kn/N}$$

$$= \cancel{\sum_{n}} \underbrace{\frac{A}{2} e^{j\phi} \sum_{n=0}^{N-1} e^{-j2\pi n(k/N - \hat{f})}}_{*} + \underbrace{\frac{A}{2} e^{-j\phi} \sum_{n=0}^{N-1} e^{-j2\pi n(k/N + \hat{f})}}_{\square}$$

Case 1 : $\hat{f} \cdot N = l$ = an integer

Recall $$\sum_{n=0}^{N-1} \alpha^{n} = \begin{cases} N & \text{if } \alpha = 1 \\ \dfrac{1-\alpha^N}{1-\alpha} & \text{if } \alpha \neq 1 \end{cases}$$

use this w/ $\alpha = e^{-j2\pi(k/N - \hat{f})}$ in (*)

$$(*) = \begin{cases} N & \text{if } \alpha = 1 \Leftrightarrow k/N - \hat{f} = 0 \Leftrightarrow k = N\hat{f} = l \\ \dfrac{1 - e^{-j2\pi \overbrace{(k/N - \hat{f})N}^{k - N\hat{f}}}}{1 - e^{-j2\pi(k/N - \hat{f})}} & \text{if } \alpha \neq 1 \Leftrightarrow k \neq l \end{cases}$$

$$= \begin{cases} N & \text{if } k = l \\ 0 & \text{if } k \neq l \end{cases}$$

need to evaluate $1 - e^{-j2\pi(k/N - \hat{f})N}$

$$= 1 - e^{-j2\pi(k - N\hat{f})}$$

$$= 1 - e^{-j2\pi(k - \ell)}$$

k integer
ℓ integer
$\Rightarrow k - \ell = \text{integer}$

$$= 1 - 1 = 0$$

$$\boxed{\ } = \begin{cases} N & \text{if } k = N - \ell \\ 0 & \text{if } k \neq N - \ell \end{cases}$$

(use geometric series with $\alpha = e^{-j2\pi(k+\ell)/N}$

$= e^{-j2\pi(k - \underline{N} + \ell)/N}$

II $\displaystyle\sum_{n=0}^{N-1} \underbrace{e^{-j2\pi n (k/N + \hat{f})}}_{\alpha^{n}}$

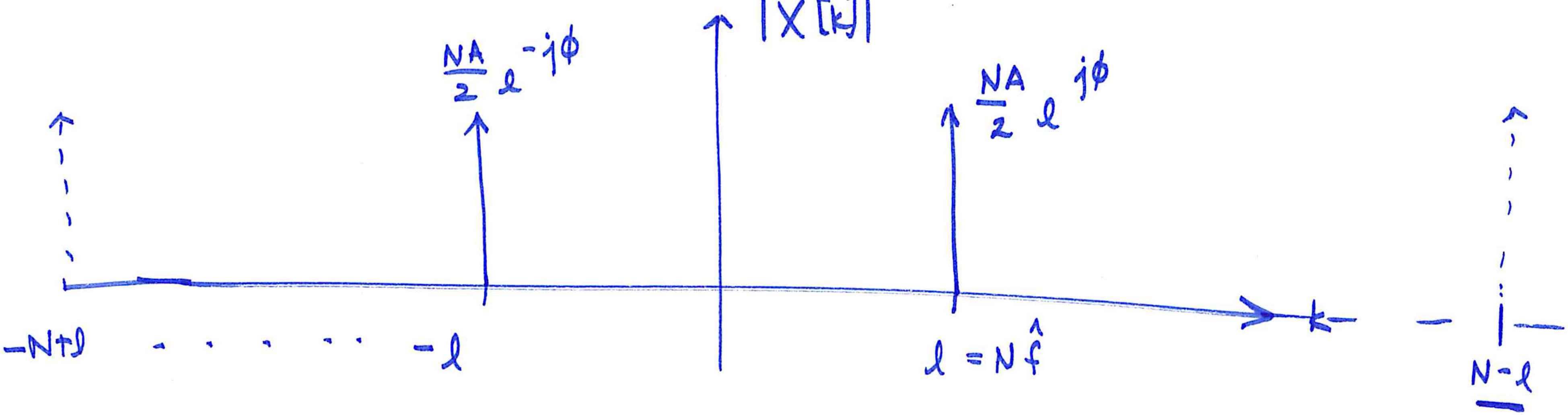
let $\alpha = e^{-j2\pi (k/N + \hat{f})}$

$= e^{-j2\pi (k/N + \hat{f} - 1)}$ $\hookrightarrow$ 2π phase shift

$$= \begin{cases} N & \text{if } \alpha = 1 \iff \underline{k/N + \hat{f} - 1} = 0 \iff k = N - N\hat{f} = \cancel{N(1-\hat{f})} \\ & \phantom{\text{if } \alpha = 1 \iff k/N + \hat{f} - 1 = 0 \iff k} = N - \ell \\ 0 & \text{if } \alpha \neq 1 \end{cases}$$

$$= \begin{cases} N & \text{if } k = N - \ell \\ 0 & \text{if } k \neq N - \ell \end{cases}$$

$$X[k] = \begin{cases} \frac{A}{2} e^{j\phi} \cdot N & \text{if} \quad k = l \\ \frac{A}{2} e^{-j\phi} N & \text{if} \quad k = N-l \\ 0 & \text{otherwise} \end{cases}$$



when $\hat{f} = \frac{f_0}{f_s} = \frac{l}{N}$, then spectrum looks like original.

Case 2: $\hat{f} \cdot N \neq$ integer

$\theta = 2\pi(k/N - \hat{f})N$

$$X[k] = \frac{A}{2} e^{j\phi} \frac{1 - e^{-j2\pi(k/N - \hat{f})\underline{N}}}{1 - e^{-j2\pi(k/N - \hat{f})1}} + \frac{A}{2} e^{-j\phi} \frac{1 - e^{-j2\pi(k/N + \hat{f})\underline{N}}}{1 - e^{-j2\pi(k/N + \hat{f})1}}$$

note for any θ, $1 - e^{-j\theta} = e^{-j\theta/2} \left[\frac{e^{j\theta/2} - e^{-j\theta/2}}{2j} \right] \cdot 2j$

$$1 - e^{-j\theta} = e^{-j\theta/2} \sin(\theta/2) \cdot 2j$$

$$X[k] = \frac{A}{2} e^{j\phi} \frac{e^{-j\pi(k/N - \hat{f})\underline{N}} \sin(\pi(k/N - \hat{f})\underline{N})}{e^{-j\pi(k/N - \hat{f})} \sin(\pi(k/N - \hat{f}))}$$

$$+ \frac{A}{2} e^{-j\phi} \frac{e^{-j\pi(k/N + \hat{f})N} \sin(\pi(k/N + \hat{f})N)}{e^{-j\pi(k/N + \hat{f})} \sin(\pi(k/N + \hat{f}))}$$