Lecture 17 : Frequency Response of FIR filters

how does an fir filter respond to a sinusoidal input?

- easier to intepret filter behavior + operation
- easier to compute + interpret convolution

Impulse response

$$x[n] = \delta[n] \longrightarrow \boxed{h} \longrightarrow h[n]$$

$$\delta[n] = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}$$

Frequency response

$$x[n] = e^{j2\pi \hat{f} n} \longrightarrow \boxed{h} \longrightarrow y[n]$$

$$y[n] = \sum_{m=-\infty}^{\infty} h[m]\,x[n-m]$$ } convolution

$$= \sum_{m=-\infty}^{\infty} h[m]\cdot e^{j2\pi\hat{f}(n-m)}$$

recall $e^{j2\pi\hat{f}(n-m)} = e^{j2\pi\hat{f}n}\cdot e^{-j2\pi\hat{f}m}$

$$= \underbrace{e^{j2\pi\hat{f}n}}_{x[n]} \underbrace{\sum_{m=-\infty}^{\infty} h[m]\,e^{-j2\pi\hat{f}m}}_{\text{called the "frequency response"}}$$

Frequency Response $H(\hat{f}) = \sum_{m=-\infty}^{\infty} h[m]\,e^{-j2\pi\hat{f}m}$

if $x[n] = e^{j2\pi\hat{f}n} \rightarrow \boxed{h} \rightarrow y[n] = e^{j2\pi\hat{f}n}\cdot H(\hat{f})$

Example: $x[n] = A\cos(2\pi\hat{f}n + \phi) \longrightarrow \boxed{h} \longrightarrow y[n] = ?$

by Euler

$$\rightarrow x[n] = \frac{A}{2} e^{j\phi} e^{j2\pi\hat{f}n} + \frac{A}{2} e^{-j\phi} e^{-j2\pi\hat{f}n}$$

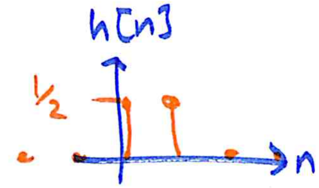
$$\Rightarrow y[n] = \frac{A}{2} e^{j\phi} e^{j2\pi\hat{f}n} H(\hat{f}) + \frac{A}{2} e^{-j\phi} e^{-j2\pi\hat{f}n} H(-\hat{f})$$

$$= A|H(\hat{f})| \cos(2\pi\hat{f}n + \phi + \angle H(\hat{f}))$$

{the filter changes the amplitude and phase of the cosine, but not its frequency}

Ex: Moving average filter

$$h[n] = \frac{1}{2}\delta[n] + \frac{1}{2}\delta[n-1]$$

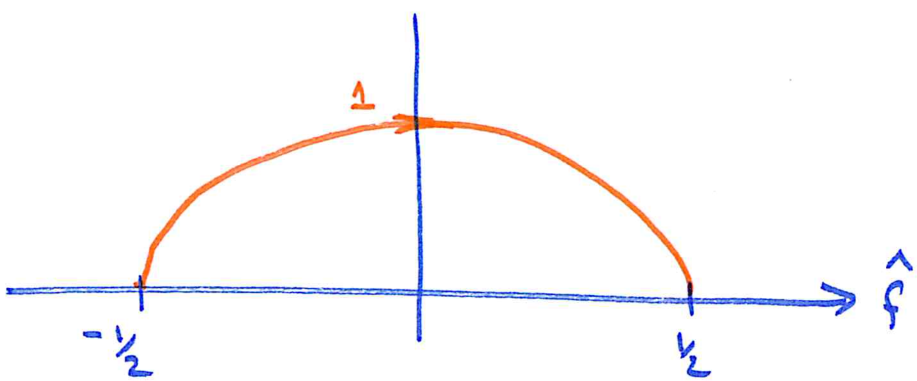


$$H(\hat{f}) = \sum_{m=\infty}^{\infty} h[m]\, e^{-j2\pi \hat{f} m}$$

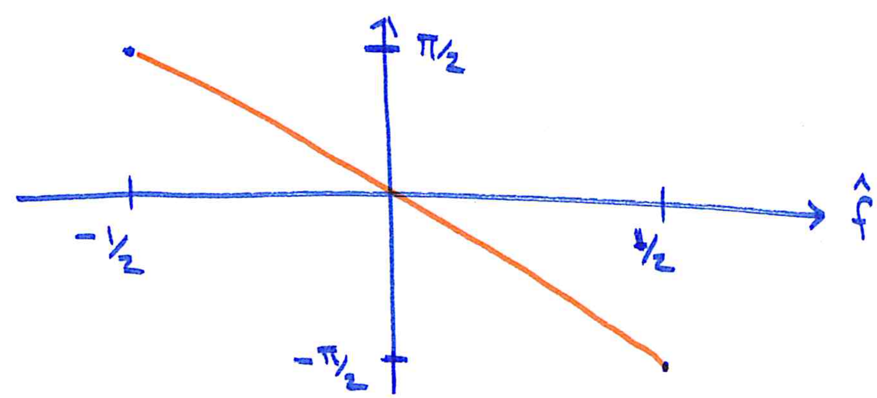
$$= \sum_{m=-\infty}^{\infty} \left(\frac{1}{2}\delta[m] + \frac{1}{2}\delta[m-1]\right) e^{-j2\pi \hat{f} m}$$

$$= \underbrace{\frac{1}{2} e^{-j2\pi \hat{f}\cdot 0}}_{=1/2} + \frac{1}{2} e^{-j2\pi \hat{f}\cdot 1} = e^{-j\pi \hat{f}} \underbrace{\left(\frac{e^{+j\pi \hat{f}}}{2} + \frac{e^{-j\pi \hat{f}}}{2}\right)}_{=\cos(\pi \hat{f})}$$

$$|H(\hat{f})| = |\cos(\pi \hat{f})|$$



$$\angle H(\hat{f}) = -\pi \hat{f}$$



- if we have input $x[n] = 2\cos(2\pi \hat{f} n)$

  · if $\hat{f}$ close to zero, $|H(\hat{f})| \approx 1$

  $\Rightarrow$ output $\approx$ input

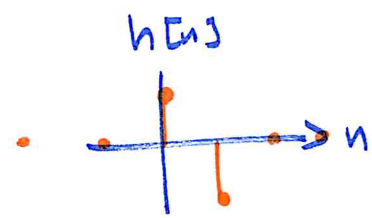
  · if $|\hat{f}| \approx 1/2$, $|H(\hat{f})| \approx 0$

  $\Rightarrow$ output ≈ 0

"low pass filter" b/c low frequency signals pass through (LPF)

Ex: digital differentiation

$$h[n] = \frac{\delta[n]}{2} - \frac{\delta[n-1]}{2}$$

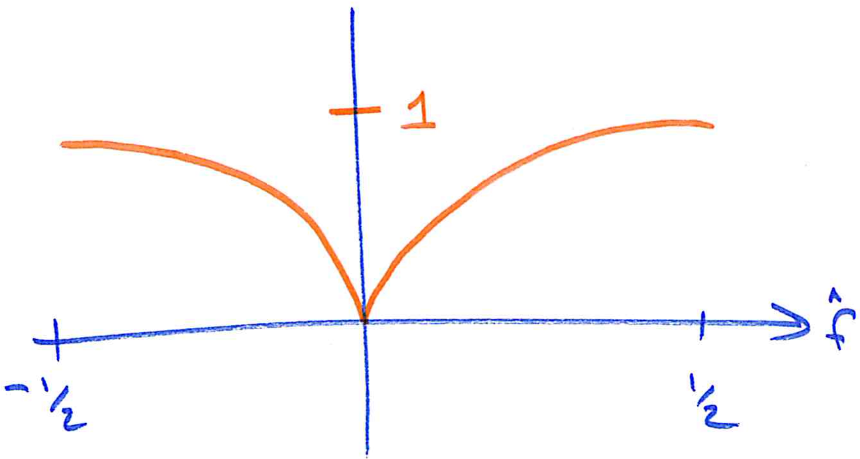


$$H(\hat{f}) = \sum_{m=-\infty}^{\infty} h[m] \cdot e^{-j2\pi \hat{f} m}$$

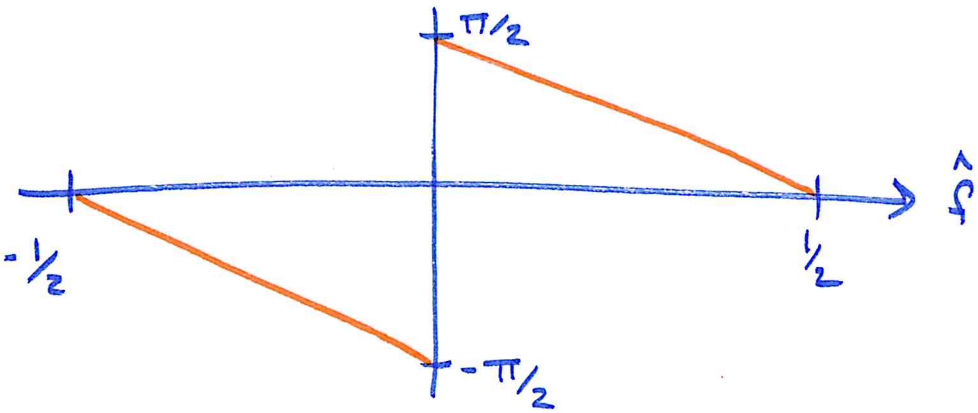
$$= \frac{1}{2} - \frac{1}{2} e^{-j2\pi \hat{f}} = \underbrace{j}_{=e^{j\pi/2}} e^{-j\pi \hat{f}} \underbrace{\left(\frac{e^{+j\pi \hat{f}}}{2j} - \frac{e^{-j\pi \hat{f}}}{2j} \right)}_{= \sin(\pi \hat{f})}$$

$$H(\hat{f}) = e^{j\pi\left(\frac{1}{2} - \hat{f}\right)} \sin(\pi \hat{f})$$

$$|H(\hat{f})| = |\sin(\pi \hat{f})|$$



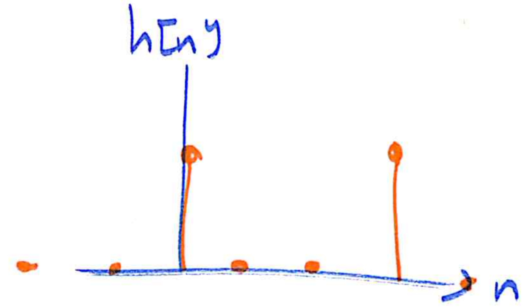
$\angle H(\hat{f})$



- low frequencies have small $|H(\hat{f})|$ → attenuated
- high frequencies are passed through

⇒ High-pass filter (HPF)

Ex: $h[n] = \delta[n] + \delta[n-3]$

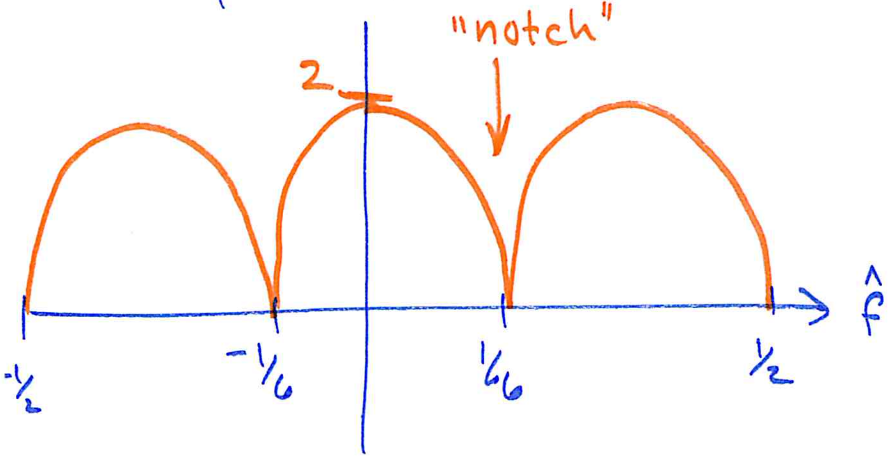


$$H(\hat{f}) = \sum_{m=-\infty}^{\infty} \underline{h[m]}\, e^{-j2\pi \hat{f} m}$$

$$= 1 + e^{-j2\pi \hat{f} \cdot \underline{3}} = e^{-j3\pi \hat{f}} \left(e^{+j3\pi \hat{f}} + e^{-j3\pi \hat{f}} \right)$$

$$= 2 e^{-j3\pi \hat{f}} \cos(3\pi \hat{f})$$

$$|H(\hat{f})| = 2 |\cos(3\pi \hat{f})|$$



"notch filter"