Lecture 18

Response of filters

impulse response

$$\delta[n] \longrightarrow \boxed{h} \longrightarrow h[n]$$

general input

$$x[n] \longrightarrow \boxed{h} \longrightarrow y[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

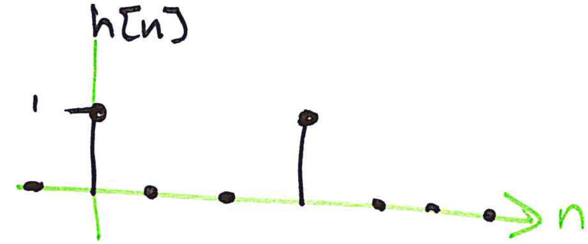
frequency response

$$e^{j2\pi \hat{f} n} \longrightarrow \boxed{h} \longrightarrow \sum_{k=-\infty}^{\infty} h[k]\, e^{j2\pi \hat{f}(n-k)}$$

$$= \underbrace{e^{j2\pi \hat{f} n}}_{\text{input}} \underbrace{\sum_{k=-\infty}^{\infty} h[k]\, e^{-j2\pi \hat{f} k}}_{\circledast\ H(\hat{f}) = \text{frequency response}}$$

(2)

Ex: $h[n] = \delta[n] + \delta[n-3]$



$$H(\hat{f}) = \sum_{k=-\infty}^{\infty} h[k] e^{-j2\pi \hat{f} k}$$

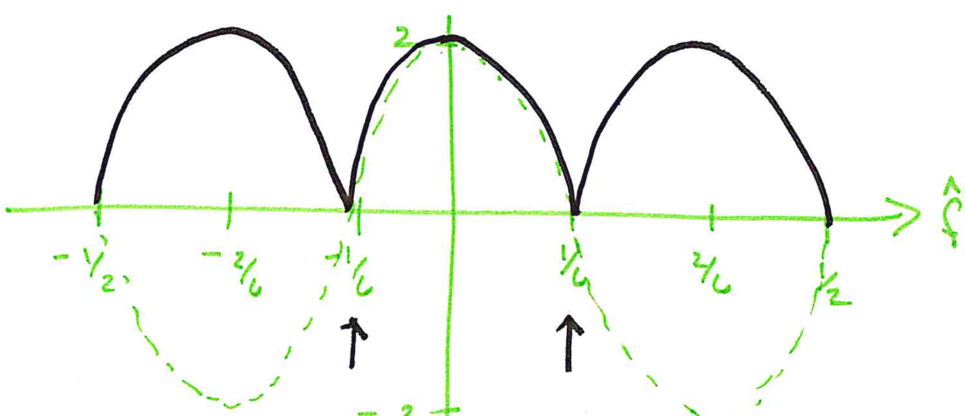
$$1 + a^2 = a\left(\underline{\tfrac{1}{a} + a}\right)$$

$$= \sum_{k=-\infty}^{\infty} \left(\delta[k] + \delta[k-3]\right) e^{-j2\pi \hat{f} k}$$

$$= 1 + \underbrace{e^{-j2\pi \hat{f}\cdot 3}}_{a^2} = \underbrace{e^{-j\pi \hat{f} 3}}_{a}\left(\underbrace{e^{+j\pi \hat{f} 3}}_{1/a} + \underbrace{e^{-j\pi \hat{f} 3}}_{a}\right)$$

$$= \boxed{2 e^{-j\pi \hat{f} 3} \cos(\pi \hat{f} 3)}$$

$$|H(\hat{f})| = 2\left|\cos(\pi \hat{f} 3)\right|$$



to get $\angle H(\hat{f})$

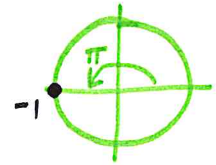
- $\hat{f} \in [-1/6, 1/6]$: $\cos(\pi \hat{f}\cdot 3) \geq 0$

  $\Rightarrow \angle H(\hat{f}) = -\pi \hat{f}\cdot 3$

- $\hat{f} \in (1/6, 1/2]$: $\cos(\pi \hat{f}\cdot 3) < 0$

  $H(\hat{f}) = \underbrace{\left(-2\cos(\pi \hat{f} 3)\right)}_{>0}\left(-e^{-j\pi \hat{f} 3}\right)$

recall $-1 = e^{j\pi}$



complex number

$Ae^{j\phi}$

where $A \geq 0$

$$-e^{-j\pi \hat{f}3} = e^{j\pi} e^{-j\pi \hat{f}3}$$

$$= e^{j\pi(1-\hat{f}3)}$$

$$\Rightarrow \quad \angle H(\hat{f}) = \underline{(1-\hat{f}3)\pi} \text{ for } \hat{f} \in (1/6, 1/2]$$

$\cdot$ $\hat{f} \in [-1/2, -1/6) \Rightarrow \cos(\pi \hat{f} \cdot 3) < 0$

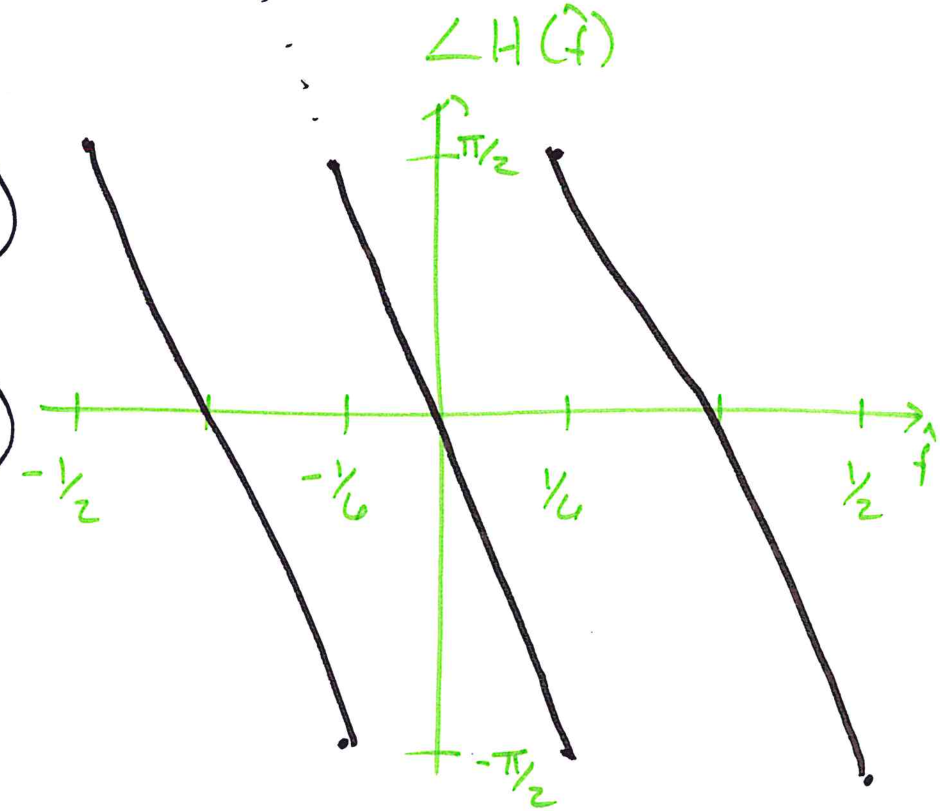
$$H(\hat{f}) = \underbrace{(-2\cos(\pi \hat{f}3))}_{>0} \left(-e^{-j\pi \hat{f}3}\right)$$

$$= (-2\cos(\pi \hat{f}3))\left(e^{j\pi(1-\hat{f}3)}\right)$$

$$\Rightarrow \quad \angle H(\hat{f}) = (1-\hat{f}3)\pi$$

@ $\hat{f} = -1/2$, $\angle H(\hat{f}) = (1+3/2)\pi$

$$= \frac{5\pi}{2} = 2\pi + \frac{\pi}{2}$$



matlab: plot(fhat, angle(H))

In general:

if input is $x[n] = a_1 e^{j2\pi \hat{f}_1 n} + a_2 e^{j2\pi \hat{f}_2 n} + a_3 e^{j2\pi \hat{f}_3 n} + \dots$

then output is $y[n] = H(\hat{f}_1) a_1 e^{j2\pi \hat{f}_1 n} + H(\hat{f}_2) a_2 e^{j2\pi \hat{f}_2 n} + H(\hat{f}_3) a_3 e \dots$

$H(\hat{f})$ is a complex number that reflects how a filter changes the magnitude and phase of an input sinusoid of frequency $\hat{f}$.

$$H(\hat{f}) = |H(\hat{f})| e^{j\angle H(\hat{f})}$$

From frequency response, we can impulse response:

$$H(\hat{f}) = \sum_{k=-\infty}^{\infty} h[k] e^{-j2\pi \hat{f} k}$$

so if $H(\hat{f}) = \underline{b_0} + \underline{b_1} e^{-j2\pi \hat{f}\cdot 1} + \underline{b_2} e^{-j2\pi \hat{f}\cdot 2} + \underline{b_3} e^{-j2\pi \hat{f}\cdot 3} + \dots$

then $\underline{h[n] = b_n}$

what if input $x[n] = 1 + \cos(2\pi \frac{1}{6} n)$?

$$= 1 \cdot e^{j2\pi 0 \cdot n} + \frac{1}{2} e^{j2\pi \frac{1}{6} n} + \frac{1}{2} e^{-j2\pi \frac{1}{6} n}$$

$$y[n] = 1 \cdot e^{j2\pi 0 \cdot n} H(\hat{f}=0) + \frac{1}{2} e^{j2\pi \frac{1}{6} n} H(\hat{f}=1/6) + \frac{1}{2} e^{-j2\pi \frac{1}{6} n} \cdot H(\hat{f}=-1/6)$$

$H(\hat{f}=0) = 2$

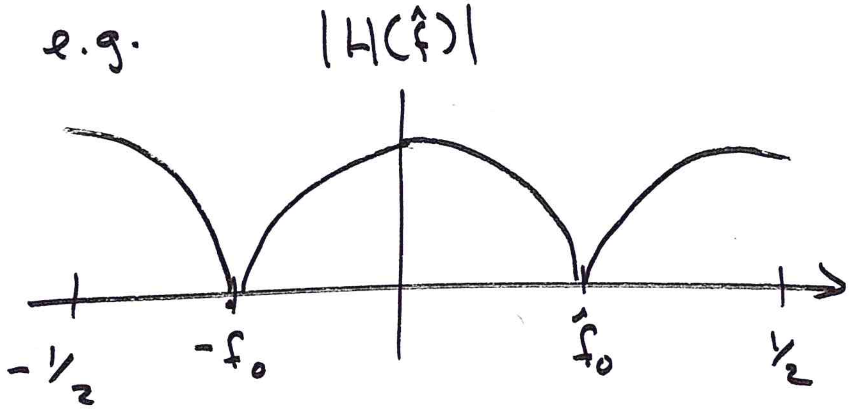
$H(\hat{f}=1/6) = 0$ $\Rightarrow y[n] = 2$

$H(\hat{f}=-1/6) = 0$

Filter design

Let's design a filter to remove a particular frequency $\hat{f}_0$

e.g.



One approach: set $H(\hat{f}) = \left(e^{-j2\pi\hat{f}} - e^{-j2\pi\hat{f}_0}\right)\left(e^{-j2\pi\hat{f}} - e^{+j2\pi\hat{f}_0}\right)$

$$= e^{-j4\pi\hat{f}} - e^{-j2\pi(\hat{f}-\hat{f}_0)} - e^{-j2\pi(\hat{f}+\hat{f}_0)} + 1$$

$$= e^{-j4\pi\hat{f}} - e^{-j2\pi\hat{f}}\left(\frac{e^{j2\pi\hat{f}_0} + e^{-j2\pi\hat{f}_0}}{2}\right)\cdot 2 + 1$$

$$= \underbrace{e^{-j4\pi\hat{f}}}_{e^{-j2\pi\hat{f}k} \text{ w/ } k=2} - \underbrace{e^{-j2\pi\hat{f}}}_{e^{-j2\pi\hat{f}k} \text{ w/ } k=1} \cdot \cos(2\pi\hat{f}_0)\cdot 2 + \underbrace{1}_{e^{-j2\pi\hat{f}k} \text{ w/ } k=0}$$

$$H(\hat{f}) = \underbrace{1}_{h[0]} \cdot e^{-j2\pi\hat{f}\cdot 0} \underbrace{- 2\cos(2\pi\hat{f}_0)}_{h[1]} e^{-j2\pi\hat{f}\cdot 1} + \underbrace{1}_{h[2]} \cdot e^{-j2\pi\hat{f}\cdot 2}$$

$$h[n] = \delta[n] - 2\cos(2\pi\hat{f}_0)\,\delta[n-1] + \delta[n-2]$$

Ex: $h_1[n] = \delta[n] - q\,\delta[n-1]$

$$H_1(\hat{f}) = \sum_{k=-\infty}^{\infty} h[k]\, e^{-j2\pi\hat{f}k} = \sum_{k=-\infty}^{\infty} \left(\delta[k] - q\,\delta[k-1]\right) e^{-j2\pi\hat{f}k}$$

$$= 1 - q\,e^{-j2\pi\hat{f}}$$

Ex : $h_2[n] = \sum_{\ell=0}^{M} r^{\ell}\,\delta[n-\ell]$

$$H_2(\hat{f}) = \sum_{k=-\infty}^{\infty}\left(\sum_{\ell=0}^{M} r^{\ell}\,\delta[k-\ell]\right)e^{-j2\pi k\hat{f}}$$

$$= \sum_{\ell=0}^{M} r^{\ell}\left(\sum_{k=-\infty}^{\infty}\delta[k-\ell]\,e^{-j2\pi k\hat{f}}\right)$$

$$= \sum_{\ell=0}^{M} r^{\ell} e^{-j2\pi\ell\hat{f}} = \sum_{\ell=0}^{M}\left(re^{-j2\pi\hat{f}}\right)^{\ell}$$

$$= \frac{1-\left(re^{-j2\pi\hat{f}}\right)^{M+1}}{1-re^{-j2\pi\hat{f}}}$$