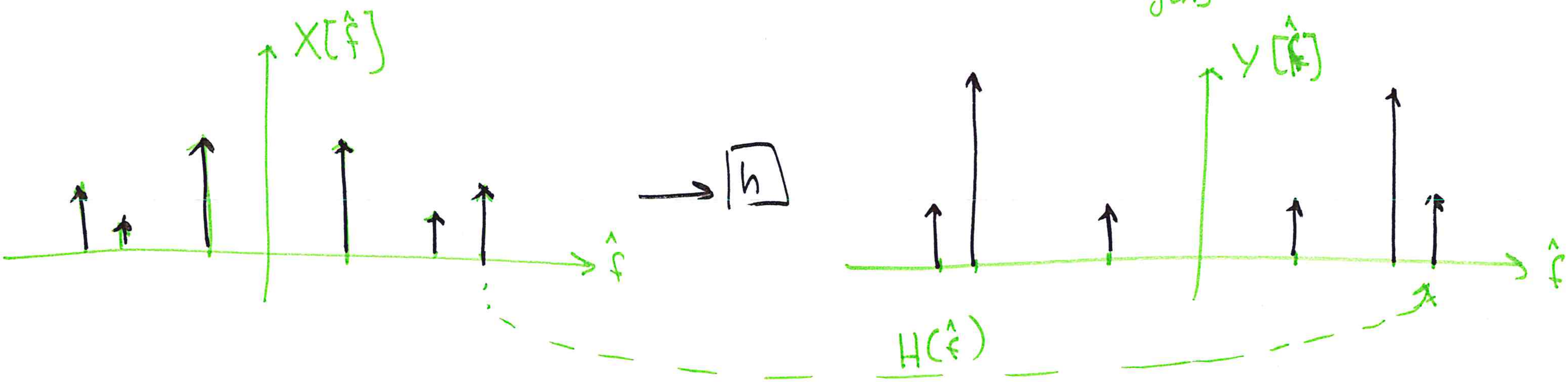
Lecture 19: Spectrum of Filter Output

Filter frequency response

$$e^{j2\pi \hat{f} n} \longrightarrow \boxed{h} \longrightarrow \underline{H(\hat{f})} \cdot e^{j2\pi \hat{f} n}$$

If input is sum of sinusoids

$$\underbrace{a_1 e^{j2\pi \hat{f}_1 n} + a_2 e^{j2\pi \hat{f}_2 n}}_{x[n]} + \dots \longrightarrow \boxed{h} \longrightarrow \underbrace{a_1 \underline{H(\hat{f}_1)} e^{j2\pi \hat{f}_1 n} + a_2 \underline{H(\hat{f}_2)} e^{j2\pi \hat{f}_2 n}}_{y[n]} + \dots$$



spectrum of output @ frequency $\hat{f}$: $\boxed{Y(\hat{f}) = X(\hat{f}) \cdot H(\hat{f})}$

Two way to calculate filter output:

1) convolution in time domain: $y[n] = x[n] * h[n]$

2) multiplication if frequency domain

Spectrum of $(x[n] * h[n]) = X(\hat{f}) \cdot H(\hat{f})$

$\boxed{\text{convolution in time domain} \Longleftrightarrow \text{multiplication in frequency domain}}$

DFT expansion of a signal with N samples:

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j2\pi \frac{k}{N} \cdot n}$$

$\Rightarrow$ expressing signal as a sum of sinusoids w/ frequencies $0, \pm 1/N, \pm 2/N, \ldots, \frac{1}{2}$

$$x[n] \longrightarrow \boxed{h} \longrightarrow \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j2\pi \frac{k}{N} n} \cdot \boxed{H\left(\hat{f} = \frac{k}{N}\right)}$$

Recall $H(\hat{f}) = \sum_{l=-\infty}^{\infty} h[l] e^{-j2\pi l \hat{f}}$ = frequency response

$$H[k] = \sum_{l=0}^{N-1} h[l] e^{-j2\pi \frac{k}{N} l}$$ = DFT of filter impulse response

$$H\left(\hat{f} = \frac{k}{N}\right) = H[k]$$

$$= \sum_{l=-\infty}^{\infty} h[l]\, e^{-j2\pi l \frac{k}{N}} = \sum_{l=0}^{N-1} h[l]\, e^{-j2\pi l k/N}$$

if $h[l] = 0$ for $l < 0$ and $l \geq N$

$$= H[k] = \text{DFT of } h$$

back to ∗

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, e^{j2\pi kn/N} \longrightarrow \boxed{h} \longrightarrow y[n] = \frac{1}{N}\sum_{k=0}^{N-1} \underbrace{X[k]\, H[k]}_{\text{multiplication in frequency domain}} e^{j2\pi kn/N}$$

Filter cascade:

$$x[n] \longrightarrow \boxed{h_1} \longrightarrow \boxed{h_2} \longrightarrow y[n] = x[n] * h_1[n] * h_2[n]$$

equivalent to

$$x[n] \longrightarrow \boxed{h_1[n] * h_2[n]} \longrightarrow y[n] =$$

Using the spectrum of filter outputs:

$$\left[X(\hat{f}) \cdot H_1(\hat{f})\right] \cdot H_2(\hat{f}) = X(\hat{f}) \cdot \left[H_1(\hat{f})\, H_2(\hat{f})\right]$$

To compute filter output:

1. DFT of $x[n] \longrightarrow X[k]$
2. DFT of $h[n] \longrightarrow H[k]$ (use zero padding)

   $h[n]$ and add zeros onto end to make it length-N
3. Multiply $X[k]H[k] \longrightarrow Y[k]$
4. IDFT of $Y[k] \longrightarrow y[n]$

in matlab:

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N = length(x);
M = length(h);
h = [h, zeros(1, N-M)];
conv(x,h) ≈ ifft(fft(x) .* fft(h))
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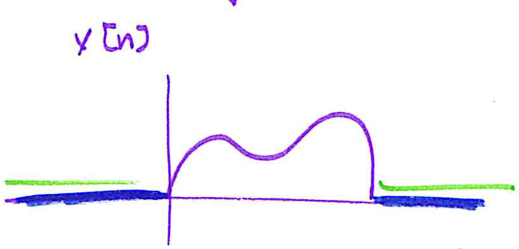
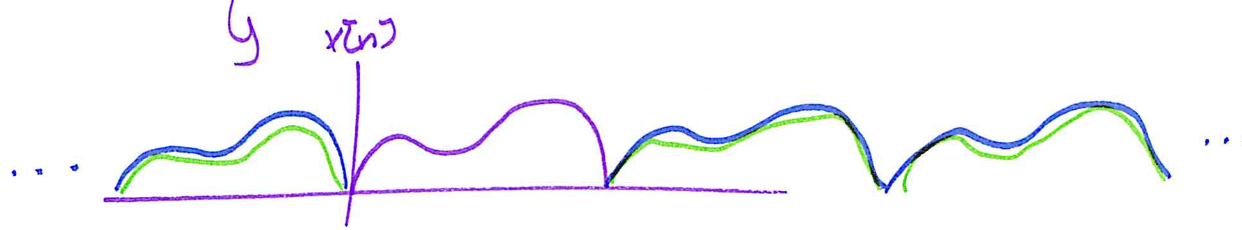
only approximately equal because of edge effects





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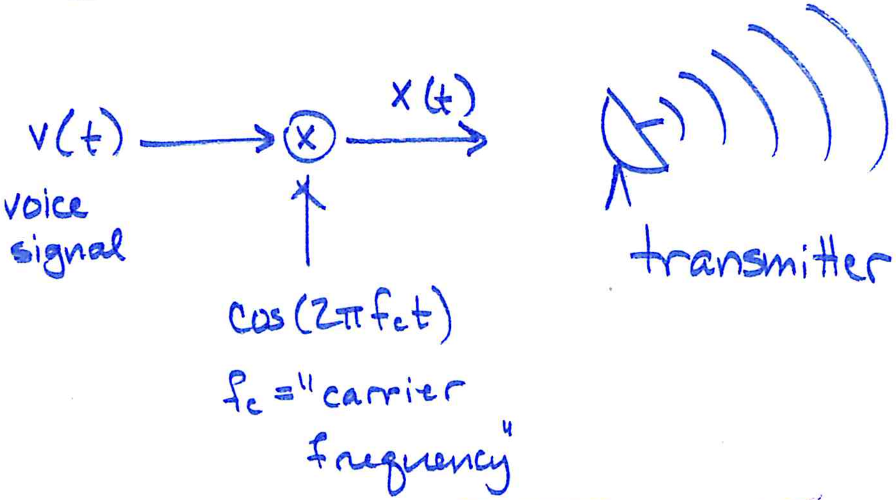
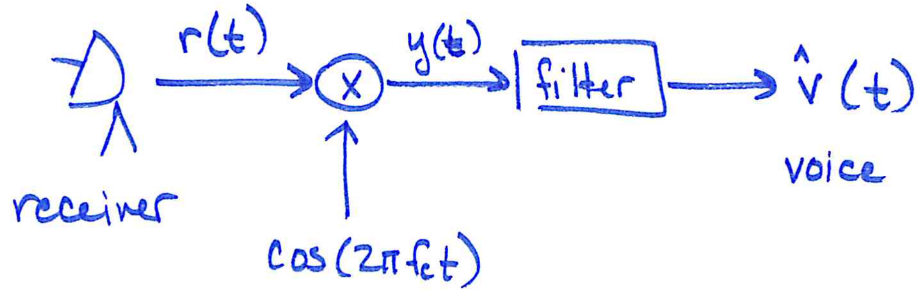


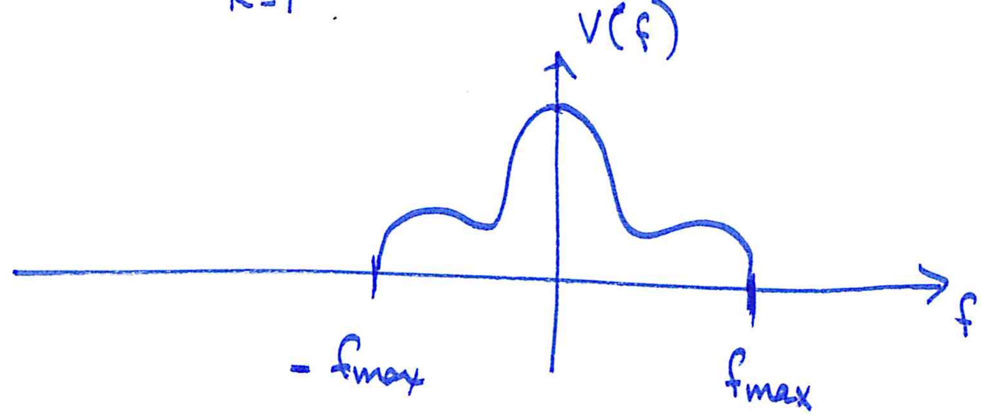


modulation: shifting spectrum of signal so it can be transmitted more easily

demodulation – extract $v(t)$ from modulated signal.

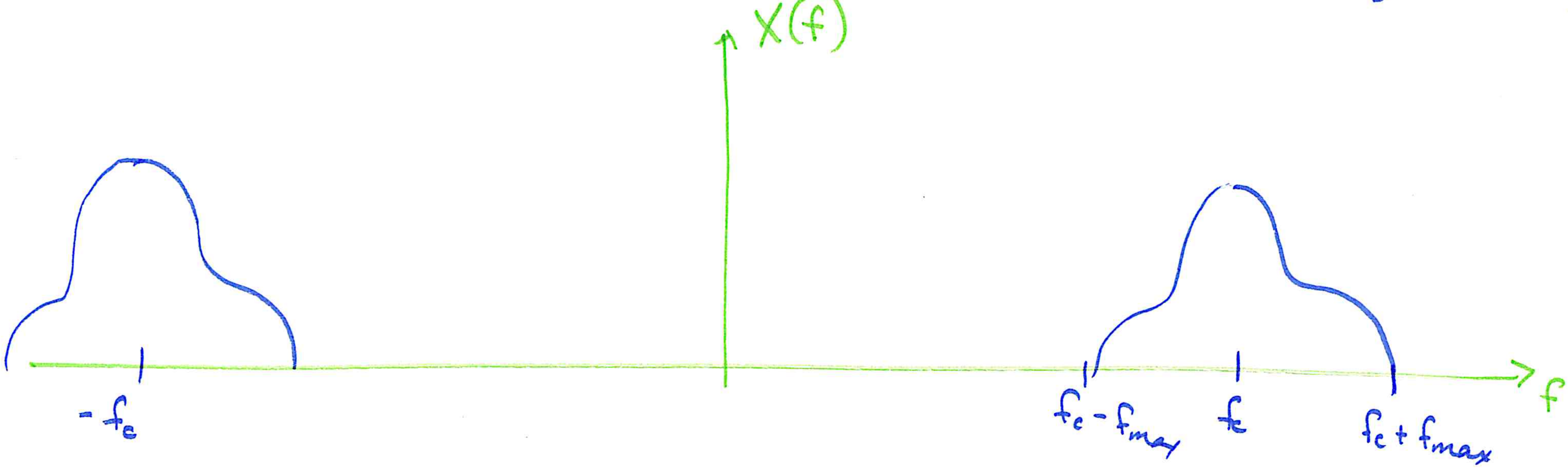
Modulation:

$$x(t) = v(t) \cdot \cos(2\pi f_c t) \qquad \text{where } v(t) = \sum_{k=1}^{N} \left(a_k e^{j2\pi f_k t} + a_k^* e^{-j2\pi f_k t} \right)$$



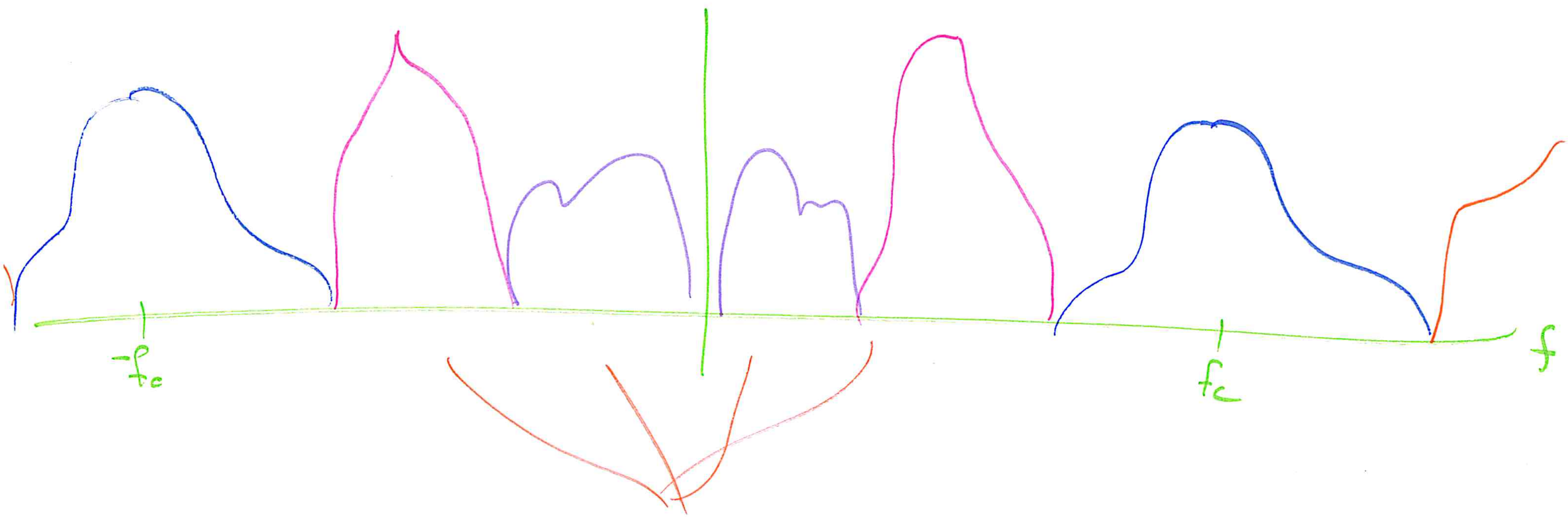
$$= \sum_{k=1}^{N} \left(a_k e^{j2\pi f_k t} + a_k^* e^{-j2\pi f_k t} \right) \cdot \left(\frac{1}{2} e^{j2\pi f_c t} + \frac{1}{2} e^{-j2\pi f_c t} \right)$$

$$= \sum_{k=1}^{N} \left[\frac{a_k}{2} e^{j2\pi (f_c + f_k) t} + \frac{a_k^*}{2} e^{j2\pi (f_c - f_k) t} + \frac{a_k^*}{2} e^{-j2\pi (f_c - f_k) t} + \frac{a_k}{2} e^{-j2\pi (f_c + f_k) t} \right]$$



$r(t)$ = received signal = transmitted signal $x(t)$ + signals from other stations

$R(f)$



$r(t) = x(t) + o.s.$
(other stations)

How to get $v(t)$?

1. multiply $r(t)$ by $\cos(2\pi f_c t)$

$$y(t) = r(t)\cos(2\pi f_c t)$$
$$= (x(t) + o.s.)\cos(2\pi f_c t)$$

$$= \left(v(t) \underline{\cos(2\pi f_c t)} + \text{o.s.} \right) \underline{\cos(2\pi f_c t)}$$

$$y(t) = \frac{1}{2} v(t) + \underline{\frac{1}{2} v(t) \cos(4\pi f_c t)}$$

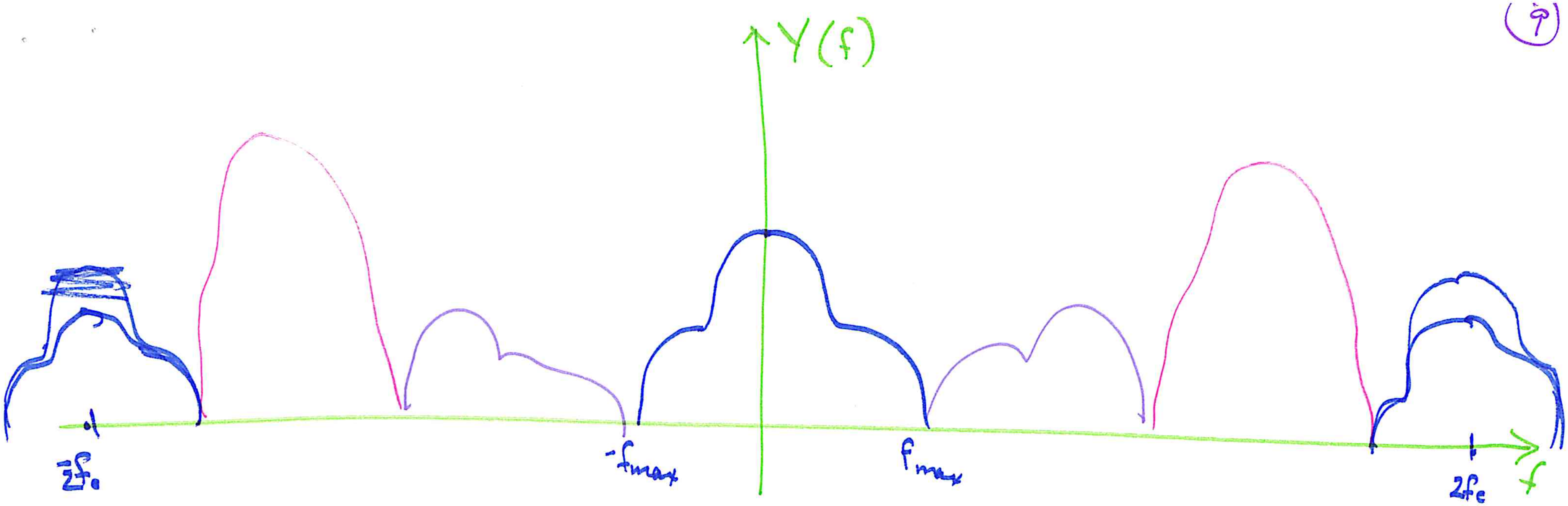
$$+ \text{o.s.} \cdot \cos(2\pi f_c t)$$

$$\cos(2\pi f_c t)^2$$

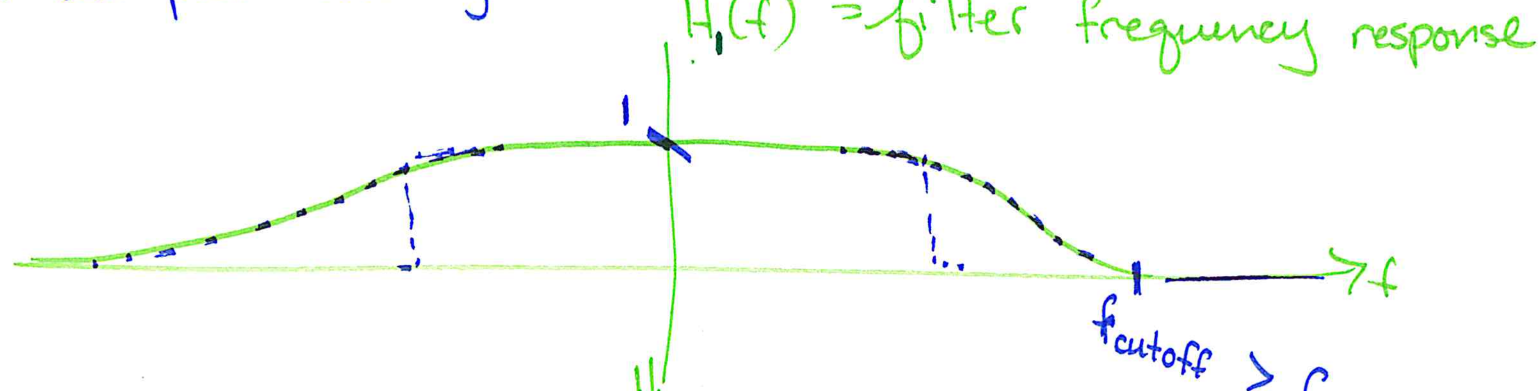
$$= \left(\frac{1}{2} e^{j2\pi f_c t} + \frac{1}{2} e^{-j2\pi f_c t} \right)^2$$

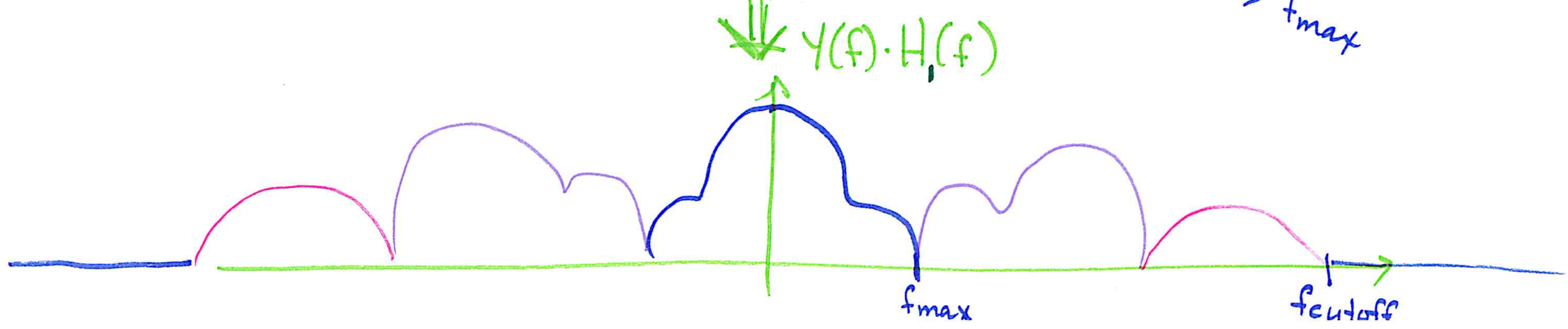
$$= \underbrace{\frac{1}{4} e^{j4\pi f_c t} + \frac{1}{4} e^{-j4\pi f_c t}} + \frac{1}{2}$$

$$= \frac{1}{2} \cos(4\pi f_c t) + \frac{1}{2}$$



2. Apply a low pass analog filter

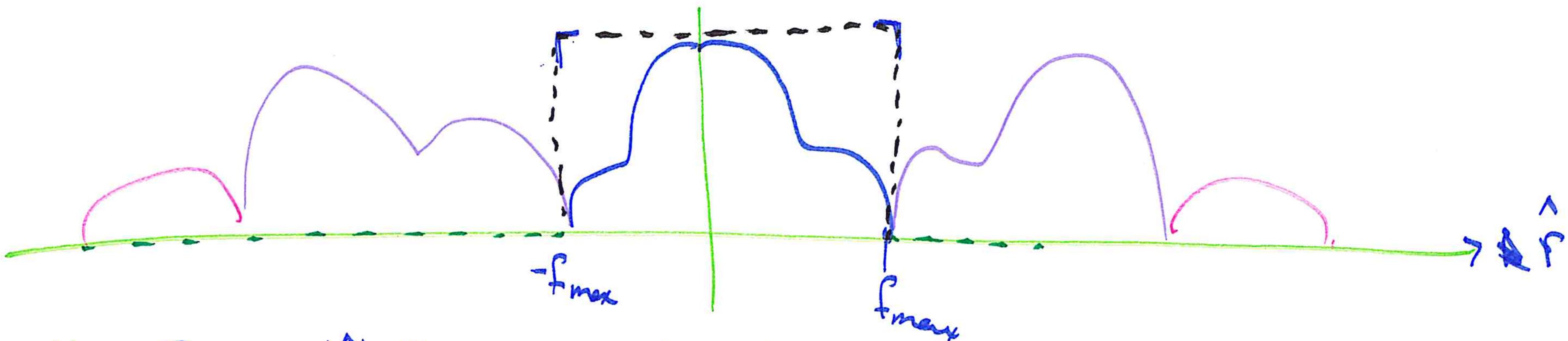




3. Sample with an A/D converter @ frequency $f_s \geq 2 f_{cutoff}$

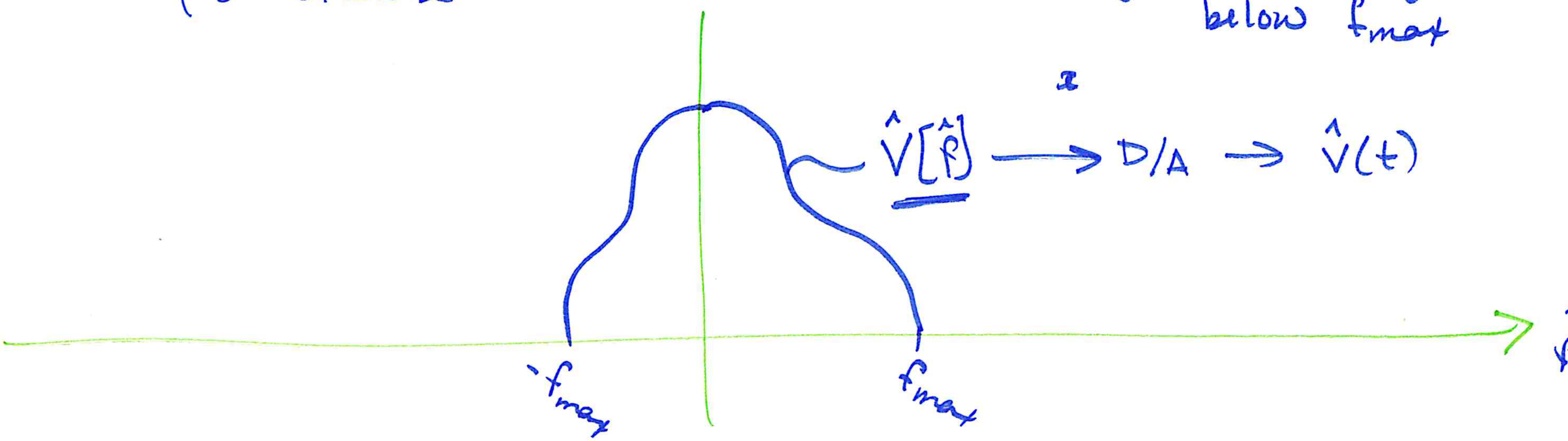
4. Digital filter

$$Y[\hat{f}]H_1[\hat{f}]$$



$$H_2(\hat{f}) = \begin{cases} 1 & \text{if } |\hat{f}| < \frac{f_{max}}{f_s} \\ 0 & \text{otherwise} \end{cases}$$

$\left(Y[\hat{f}]H_1[\hat{f}]\right)H_2[\hat{f}]$ → only "passes" frequencies below f_{max}



$\hat{V}[\hat{f}] \longrightarrow D/A \longrightarrow \hat{V}(t)$