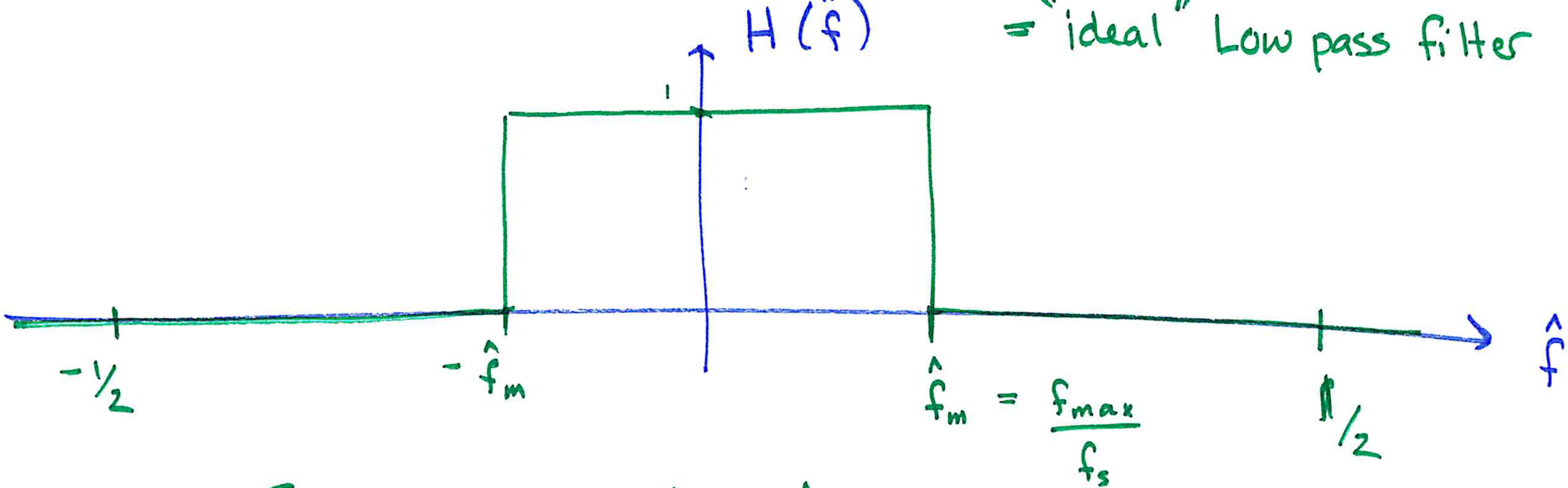
Lecture 20: FIR filter design

$H(\hat{f})$ = "ideal" Low pass filter



$$H(\hat{f}) = \begin{cases} 1 & \text{if } |\hat{f}| \le \hat{f}_m \\ 0 & \text{otherwise} \end{cases}$$

Recall: $$H(\hat{f}) = \sum_{k=-\infty}^{\infty} h[k]\, e^{-j 2\pi \hat{f} k} \quad (*)$$

So are the $h[k]$'s ?

Same math as with Fourier Series!

Fourier Series

$$x(t) = \sum_{k=-\infty}^{\infty} a_k \, e^{j2\pi f_0 t k}$$

where

$$a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j2\pi f_0 t k} \, dt$$

$$= \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) e^{-j2\pi f_0 t k} \, dt$$

$t \to \hat{f}$

$f_0 = -1$

$a_k \to h[k]$

$x(t) \to H(\hat{f})$

$$H(\hat{f}) = \sum_{k=-\infty}^{\infty} h[k] \, e^{-j2\pi \hat{f} k} \quad (*)$$

$$h[k] = \int_{-1/2}^{1/2} H(\hat{f}) e^{j2\pi \hat{f} k} \, d\hat{f}$$

$$h[k] = \int_{-1/2}^{1/2} H(\hat{f})\, e^{j2\pi \hat{f} k}\, d\hat{f}$$

$$= \int_{-\hat{f}_m}^{\hat{f}_m} 1 \cdot e^{j2\pi \hat{f} k}\, d\hat{f}$$

if $k = 0$

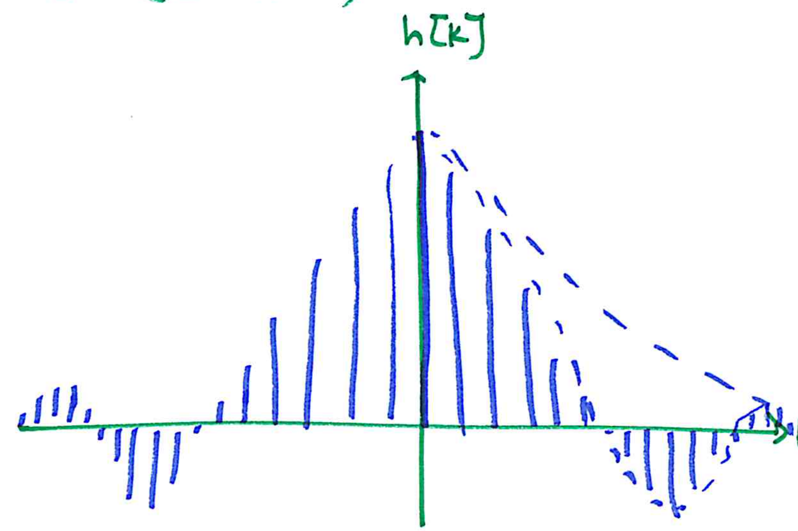
$$h[0] = \int_{-\hat{f}_m}^{\hat{f}_m} 1\, d\hat{f} = 2\hat{f}_m$$

if $k \neq 0$

$$h[k] = \frac{1}{j2\pi k}\, e^{j2\pi \hat{f} k}\Big|_{-\hat{f}_m}^{\hat{f}_m} = \frac{1}{\pi k} \cdot \underbrace{\frac{1}{2j}\left(e^{j2\pi \hat{f}_m k} - e^{-j2\pi \hat{f}_m k}\right)}_{\sin(2\pi \hat{f}_m k)}$$

$$\boxed{h[k] = \begin{cases} \dfrac{\sin(2\pi \hat{f}_m k)}{\pi k} & \text{if } k \neq 0 \\[2ex] 2\hat{f}_m & \text{if } k = 0 \end{cases}}$$

= "sinc" function



Problems with what we have so far:

$h[k]$ is defined for $k = 0, \pm 1, \pm 2, \pm 3, \ldots$

- not finite impulse response
- not causal

There are two ways we will discuss to address this:

1. Truncate the $h[k]$'s : "truncated sinc filter"

Idea : for large k, $h[k] \approx 0$

$$\tilde{h}_{TS}[k] = \begin{cases} h[k] & \text{if } |k| \leq L \\ 0 & \text{otherwise} \end{cases} \quad \Rightarrow \tilde{h}_{TS} \text{ is FIR}$$

next: shift filter in time

$$h_{TS}[k] = \tilde{h}_{TS}[k-L] \qquad \Rightarrow h_{TS} \text{ is FIR and causal.}$$

$$h_{TS}[k] = \begin{cases} \dfrac{\sin(2\pi \hat{f}_m (k-L))}{\pi(k-L)} & \text{for } k \neq L, \\ & k = 0, 1, 2, \ldots, 2L \\ 2\hat{f}_m & \text{for } k = L \end{cases}$$

(2) Parks - McClellan filter design

Goal: find a length $2L+1$ filter with a frequency response $H_{PM}(\hat{f})$ that minimizes error:

$$\min_{h_{PM}} \max_{-\frac{1}{2} \leq \hat{f} \leq \frac{1}{2}} \left| H_{PM}(\hat{f}) - H(\hat{f}) \right| \Leftarrow$$

minimize the error at the frequency with the worst error. Often called minimax (minimize max error)